Quantum Correlations-Entanglement and beyond Entanglement

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Main Objective of the talk

- To discuss different measures of non-classical correlations:- entanglement and as well beyond entanglement scenario.
- To begin with, we first explain some of the background materials.

Non-locality: a discovery! Or…

- The term non-locality is one of the most important and debatable word in the last century.
- If we break the whole period in some parts in the sense that due to non-local (!) behavior someone may consider quantum mechanics is not a complete theory or we may think or try to understand the word in some operational way through the results from quantum mechanics.
- For the first part we usually look for hidden variable approaches and for the second part we find some fascinating results almost counterintuitive in nature.

Understanding Quantum Correlations

- If we restrict ourselves with the results from quantum mechanics only, we find the behavior of quantum systems is not fully understood whenever there are more number of parties involved.
- In other words, there exist a peculiar type correlation between the parties involved which is not explainable by classical scenario.

Understanding......contd.

- Naturally, one may ask, how one could formalise the concept of correlations in quantum mechanical systems?
- Is quantum correlation quantifiable?
- Is there any procedure to detect such correlation?
- Or, how to characterize quantum correlations?
- All the above issues are very much fundamental in nature and they have immense importance in quantum information theory.
- There are several ways to describe correlations in composite quantum systems.

The first one we will describe is Entanglement

This is possibly the most wonderful invention of quantum mechanics.

Initially everyone thinks the correlation which is responsible for non-local behavior of quantum systems is nothing but the entanglement.

However, findings in different quantum systems show there are other candidates also.

Some Basic Notions about Quantum Systems

- Physical System- associated with a separable complex Hilbert space
- Observables are linear, self-adjoint operators acting on the Hilbert space
- States are represented by density operators acting on the Hilbert space
- Measurements are governed by two rules
- 1. Projection Postulate:- After the measurement of an observable A on a physical system represented by the state ρ, the system jumps into one of the eigenstates of A.

- 2. Born Rule:- The probability of obtaining the system in an eigen state $|\psi_i\rangle$ is given by

$$\mathrm{Tr}(\rho P[|\psi_i\rangle]).$$

- The evolution is governed by an unitary operator or in other words by Schrodinger's evolution equation.

States of a Physical System

- Suppose H be the Hilbert space associated with the physical system.
- Then by a state ρ we mean a linear, hermitian operator acting on the Hilbert space H such that
- It is non-negative definite and
- $Tr(\rho)= 1$.
- A state is pure iff $\rho^2 = \rho$ and mixed iff $\rho^2 < \rho$
- Pure state has the form $\rho=|\Psi\rangle\langle\Psi|$, $|\Psi\rangle \in H$.

Composite Systems

- Consider physical systems consist of two or more number of parties A, B, C, D, ……
- The associated Hilbert space is $H_A \otimes H_B \otimes H_C \otimes H_D \otimes \ldots$
- States are then classified in two ways
- (I) Separable:- have the form,
- $\rho_{ABCD} = \sum w_i \, \rho_i^A \otimes \rho_i^B \otimes \rho_i^C \otimes \rho_i^D$ with $0 \le w_i \le 1$,
- and $\sum w_i = 1$.
- (II) All other states are entangled.

Bipartite Pure States

- Pure bipartite states have the Schmidt decomposition form,
- $|\Psi\rangle_{AB} = \sum\sqrt{\lambda_i}\ |i\rangle_A \otimes |i\rangle_B$ where
- $\{|i\rangle_A\}$ and $\{\ |i\rangle_B\ \}$ are the Schmidt bases of the parties A and B and $\lambda_1, \lambda_2, \ldots,$ are the Schmidt coefficients that satisfies $0 \leq \lambda_i \leq 1$, and $\sum\lambda_i = 1$.
- Pure product states have only one Schmidt coefficient and entangled states have more than one.

Some Use of Quantum Entanglement

- Quantum Teleportation,(Bennett et.al., PRL, 1993)
- Dense coding, (Bennett et.al., PRL, 1992)
- Quantum cryptography, (Ekert, PRL, 1991)

Physical Operation

Suppose a physical system is described by a state ρ Krause describe the notion of a physical operation defined on ρ as a completely positive map ξ, acting on the system and described by

$$\xi(\rho) = \sum_k A_k \rho A_k^\dagger$$

where each A_k is a linear operator that satisfies the relation

$$\sum_k A_k^\dagger A_k \leq I$$

Separable Super operator

If $\sum_k A_k^{\dagger} A_k = I$ then the operation A_k is trace preserving. When the state is shared between a number of parties, say, A, B, C, D,. ....
and each A_k has the form

$$A_k = L_k^A \otimes L_k^B \otimes L_k^C \otimes L_k^D \otimes \cdots\cdots$$

with all of $L_k^A, L_k^B, L_k^C, L_k^D, \cdots\cdots$

are linear operators then the operation ξ is said to be a separable super operator.

Local operations with classical communications (LOCC)

Consider a physical system shared between a number of parties situated at distant laboratories. Then the joint operation performed on this system is said to be a LOCC if it can be achieved by a set of some local operations over the subsystems at different labs together with the communications between them through some classical channel.

Result : Every LOCC is a separable superoperator.

But whether the converse is also true or not ?

It is affirmed that there are separable superoperators which cannot be expressed by finite LOCC.

Is entanglement quantifiable?

Qualitative equivalence of different entangled states:

2 copies of

$(1/\sqrt{2})|00\rangle + (1/2)|11\rangle + (1/2)|22\rangle$

is equivalent to

3 copies of $(1/\sqrt{2})(|00\rangle + |11\rangle)$

How massive a given object is?

- Mass = lim{(no. of standard masses)/ (no. of actual objects)}

The standard in entanglement

- The Bell states

$$\left|\Phi^{+}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|00\right\rangle + \left|11\right\rangle\right), \qquad \left|\Phi^{-}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|00\right\rangle - \left|11\right\rangle\right)$$

$$\left|\Psi^{+}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|01\right\rangle + \left|10\right\rangle\right), \qquad \left|\Psi^{-}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|01\right\rangle - \left|10\right\rangle\right)$$

Pure Bipartite Entanglement

- Entanglement of pure state is uniquely measured by von Neumann entropy of its subsystems,

$$E\left(\left|\Psi\right\rangle\right) = -\sum_{i=1}^{d} \lambda_i \log_2 \lambda_i \qquad \text{-----(II)}$$

- States are locally unitarily connected if and only if they have same Schmidt vector, hence their entanglement must be equal.

Bipartite Entanglement….

- Now, as far as bipartite entanglement is concerned we have at least some knowledge how to deal with entangled states.
- For pure bipartite states entanglement is a properly quantifiable.

- However for mixed entangled states there is no unique measure of entanglement. One has to look on different ways to quantify entanglement
- Some of the measures of entanglement are distillable entanglement, entanglement cost, entanglement of formation, relative entropy of entanglement, logarithmic negativity, squashed entanglement, etc.

Difficulty

- In most of the cases it is really hard to calculate exactly the measures of entanglement. Only for some few classes of states, actual values are available.
- A similar problem is that it is hard to find whether a mixed bipartite state is entangled or not.

Some Comments

- There are several key issues when we are dealing with entanglement.
- Whether entanglement dynamics is reversible or not?
- In other words, the amount required to create an entangled state is equal to the amount extracted from it or not?
- If we consider only LOCC then the answer is negative. Even if we consider PPT(positive under partial transposition) operations, then also the answer is negative.

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- However, if we consider asymptotically non-entangling operations, the answer is positive.
- All the issues are in asymptotic region, i.e., whenever infinite copies are available and if we consider any bipartite states, pure or mixed.
- However, for pure bipartite states entanglement dynamics is reversible under LOCC.
- Another important aspect in entanglement theory is the concept of bound entanglement, like bound energy in general physics.
- Actually, existence of bound entangled states provide us the irreversibility in entanglement dynamics under LOCC.

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- By bound entangled states we mean states with zero distillable entanglement. i.e., no entanglement could be extracted from the states under LOCC.
- There exist PPT bound entangled states, however, the question of existence of NPT bound entanglement is still a unsolved problem.
- A quite related problem from mathematical point of view is the characterization of positive maps.
- One must aware of the fact that:
- Thermo-dynamical law of Entanglement : Amount of Entanglement of a state cannot be increased by any LOCC.

Some other issues:

Local conversion of States:

- Given a pure/mixed entangled state our aim is to convert it to another specified/required state by LOCC with certainty or with some probability (SLOCC).

Local-distinguishibility/indistinguishibility of set of states, entangled or product.

- e.g.,The local-indistinguishibility of a complete set of orthonormal product states in 3x3 system. (non-locality without entanglement)

Multipartite Entanglement

- But the situation in multipartite case is more complex than that of bipartite case. e.g., how could we define a measure of entanglement for multipartite states at least for pure states are concerned. It is also very difficult to define maximally entangled states in multipartite systems.

- Consider a mixed entangled state in a multipartite system with the property that it has maximal entanglement w.r.t. any bipartite cut (i.e., reduced density matrices corresponding to the cut is proportional to the identity operator), then we observe that for n-qubit ($n \geq 3$) system, there does not exist any maximally entangled states for $n=4$ and $n \geq 8$.

Therefore one has to think how to define maximally entangled states for such situations. Gour and others have defined maximally entangled states in 4-qubit system considering some operational interpretation.

A possible way: the average bipartite entanglement w.r.t. all possible bipartite cuts the state is maximal.

Depending upon different entanglement measures, such as, tangle, Tasllis and Renyi α-entropies one could find different states which are maximally entangled w.r.t. the entangled measures considered.

Another attempt to quantify entanglement of a multipartite state, through the distance measures. e.g., geometric measure.

How to deal with multipartite entanglement!

- Recently, we find some attempts to look into the problem of quantifying multipartite entanglement. e.g., see arxiv: 1510:09164, prl, 115, 150502 (2015), prl, 111, 110502 (2013) by B. Kraus et al.
- The basic aim in all the works is to understand LOCC further to probe entanglement behaviour of composite quantum systems.
- Firstly, LOCC provides us the possible protocols with which entangled states can be manipulated.
- 2ndly, LOCC induces a operationally meaningful ordering of entangled states.

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- i.e., if the state |ψ> can be transformed to |ϕ> by LOCC, then any task that can be implemented by the later, are also amenable by using former. So |ψ> is more (or equally) useful than |ϕ> and consequently have the more (or same) amount of entanglement. (ordering possible).
- However, understanding the nature of LOCC is not always easy at all, rather it is a very difficult task in quantum information theory.
- For pure bipartite case we have definite result regarding convertibility under LOCC. We will now describe it in brief, to understand our task in multipartite case.

Local conversion of States

Basic task:

- Given a pure/mixed entangled state our aim is to convert it to another specified/required state by LOCC with certainty or with some probability (SLOCC).
- Consider here mainly pure bipartite entangled states.

- Asymptotically, it is always possible to convert any pure bipartite state to other. For mixed state it is not the case always. (irreversibility in entanglement manipulation:- think about bound entangled states.

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- Therefore, the whole strategy for pure bipartite states is on finite regime. i.e., either single copy or multiple copies of a state are given and our aim is to convert it to another by deterministic or stochastic LOCC.
- In this respect, for single copy case, Lo and Popescu(Phys. Rev. A 63, 022301 (2001)) have obtained some results:
- For entanglement manipulation of pure bipartite states 1-way communication is necessary and sufficient.
- 1-way communication is always better than no communications.
- The no. of Schmidt coefficients can never be increased under LOCC.

Nielsen's criteria

Let $|\Psi\rangle_{AB}$ and $|\Phi\rangle_{AB}$ be any two bipartite pure states of Schmidt rank d with Schmidt vectors,

$\lambda_{|\Psi\rangle} = \{\lambda_1, \lambda_2, \cdots \lambda_d\}$ and $\lambda_{|\Phi\rangle} = \{\mu_1, \mu_2, \cdots \mu_d\}$

respectively, where

$$1 \geq \lambda_i, \mu_j \geq 0 \quad \forall \quad i, j \,, \qquad \sum \lambda_i = 1 = \sum \mu_j$$ and

$$\lambda_i \geq \lambda_{i+1} \quad and \quad \mu_i \geq \mu_{i+1} \quad \forall \quad i, j = 1, 2, \cdots, d-1$$

(PRL, 83, 436 (1999))

Then the state $|\Psi\rangle_{AB}$ can be deterministically transformed to the state $|\Phi\rangle_{AB}$ by LOCC if and only if $\lambda_{|\Psi\rangle}$ majorizes $\lambda_{|\Phi\rangle}$ (denoted by $\lambda_{|\Psi\rangle} \prec \lambda_{|\Phi\rangle}$).

Are all pure states convertible by LOCC?

- Nielsen's result provide us the necessary and sufficient condition for deterministic local conversion of pure bipartite states. But all states are not convertible to each other by LOCC with certainty. e.g., consider a pair $(|\Psi\rangle, |\Phi\rangle)$ with Schmidt vectors (.4,.3,.2,.1) and (.50,.18,.16,.16). According to the above criteria the states are not locally convertible. It violets a necessary condition.
- There are several issues relating to convertibility. viz., catalysis, assistance by entanglement, SLOCC convertibility, etc.

Multipartite entanglement-contd…

- For multipartite case, convertibility by LOCC is known for only few classes of states. For this reason, we consider another two classes of transformations, LU and SLOCC.
- It provides us mathematically more tractable way and provides also operationally meaningful classification.
- We call two multipartite states are LU equivalent if there is unitary operator for each subsystem sothat one is obtained from the other.(see prl,104, 020504(2010), pra, 82, 032121(2010))
- We call two pure multipartite states are SLOCC-equivalent, if there is a locally invertible operator so

- that one in convertible to other by applying that operator.
- For three qubit system there are two SLOCC- in equivalent classes of states, viz., W and GHZ class. However, for four qubit system there are infinitely many. This is one of the major difficulty to characterize multipartite entanglement.
- To define an operationally meaningful entanglement measure, we first describe some notions: (see, prl, 115, 150502(2015)).
- We call a multipartite pure state $|\psi>$ can reach a multipartite pure state $|\phi>$, if there exists a LOCC protocol that transforms $|\psi>$ into $|\phi>$ deterministically, i.e., $|\phi>$ is accessible from $|\psi>$.

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- Consider two sets corresponding to a given state |ψ>, say, M_a(|ψ>) and M_s(|ψ>), where first one denotes set of all states accessible by LOCC from |ψ> and later denotes the set of all states that can reach |ψ>.
- Now consider two volumes Va(|ψ>) and Vs(|ψ>) corresponding to the accessible states and source states under LOCC with suitable volume measure in the set of LU equivalent classes.
- Clearly if Ms is very large, then the state is not very powerful, however if Ma is very large, then the state is definitely more valuable. This enables us to define operationally meaningful entanglement measures.

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- Now if a state is accessible from another state, then any state that can reach via LOCC to the later state will also reach the former, i.e, the set Ms of the former is contains the set Ms of other state. The reverse is for Ma.
- Therefore a possible choice would be,
- $Ea(|\psi>)=Va(|\psi>)/ Sup(Va)$ and
- $Es(|\psi>) =1- Vs(|\psi>)/Sup(Vs)$, where sup denotes the maximally accessible or source volume according to the choice measure.
- Clearly, $Ms(|\psi>)$ = null set implies $Vs =0$. and we call such states as maximally entangled states (MES).

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- In the work (prl, 115, 150502(2015)), for three qubit case, complete analysis for genuine three qubit states is given and it is found that both W and GHZ states act as MES.
- In arxiv: 1510.09164, you will find more elaborate analysis of considering the notion of multipartite entanglement and complete analysis for four qubit MES.
- The above analysis enables us to rethink the notion of multipartite entanglement again with a possible resolution. However, there are lot of things to do, e.g. what about multipartite mixed states?

An interesting issue

- One of the most interesting issue in entanglement theory is that entanglement is monogamous.
- e.g., if a pure state is shared between three parties A,B,C in such a way that two parties(say, A,B) shared a maximally entangled state then there must not be any entanglement between A with C and B with C.
- However, for other measures of non-classical correlations, it is not always the case.

Correlation measures beyond entanglement:

Consider first the measures of correlation:

Quantum Discord

H.Ollivier and W.H. Zurek, PRL, 88,017901(2001)

Quantum Discord

- Consider the following state:

$$\rho = \frac{1}{4}\left[|+\rangle\langle+| \otimes |0\rangle\langle 0| + |-\rangle\langle-| \otimes |1\rangle\langle 1| + |0\rangle\langle 0| \otimes |-\rangle\langle-| + |1\rangle\langle 1| \otimes |+\rangle\langle+| \right]$$

- The above state is separable. However, it has non-zero quantum discord which is defined by difference of measuring mutual information in two different ways, viz., D(A,B)= I(A:B)-J(A:B) where,
- I(A:B)= S(A)-S(A|B) and
- J(A:B)=S(A)-min Σ pj S(A|j)
- {Πj} j

The above quantity is a measure of non-classical correlation. It has zero value if and only if there exists a von Neumann-measurement $\Pi_k = |\Psi_k\rangle\langle\Psi_k|$ such that the bipartite state

$$\rho = \sum_k \Pi_k \otimes I \, \rho \, \Pi_k \otimes I$$

States of the above kind are known as classical-quantum state.

Some Observations

- One could interpret Discord in terms of consumption of entanglement in an extended quantum state merging protocol thus enabling it to be a measure of genuine quantum correlation.
- Physically, discord quantifies the loss of information due to the measurement.
- This correlation measure is invariant under LU but may change under other local operation. It is asymmetric w.r.t the parties.

- The set of Classical-Quantum states is non convex.
- Due to the optimization problem, it is in general very hard to find analytic expression for discord. Exact analytical result is available only for a few classes of states.
- It was found that Quantum discord is always non-negative and it reduces to Von Neumann Entropy of the reduced density matrix for pure bipartite states.

- There are other variations of discord and their extensions to multipartite systems have also been proposed. e.g.,
- Geometric discord:

$D(\rho)= \min ||\rho-\chi||$ where the minimum is taken over all zero discord state χ.

- Exact analytical formula for geometric discord is also available for only a few class of states. A tight lower bound is found recently.
- One could also define discord in terms of relative entropy: $D(\rho) = \min S(\rho||\chi)$
- The correlation measure discord actually generates the possibility of research beyond entanglement.

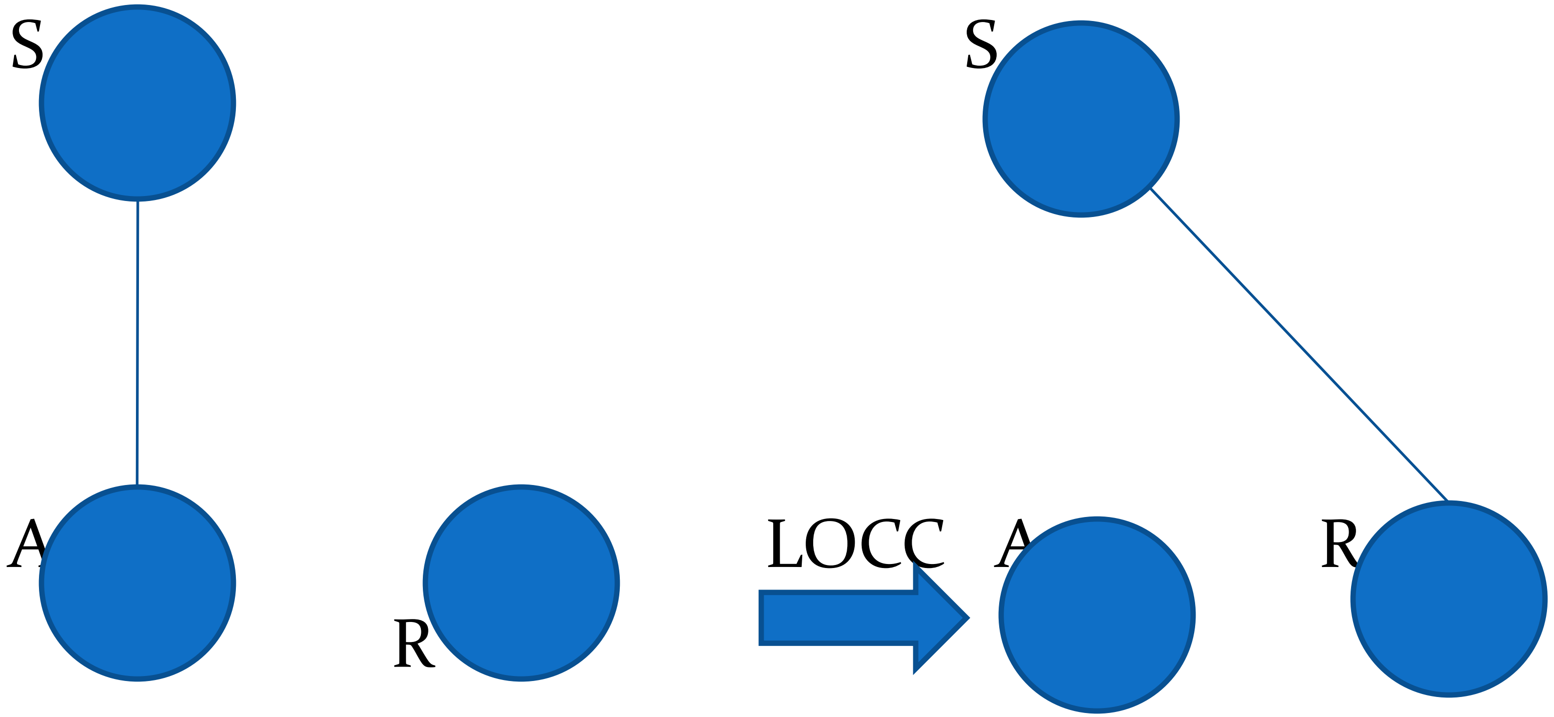
Understanding Discord!

- Dakic et al, Nature Physics, 8, 666(2012); Horodecki et al, PRL 112, 140507(2014); Giorgi, PRA 88, 022315(2013).
- Is quantum discord a useful resource for remote state preparation?
- The claim of the nature paper was: presence of discord is necessary and sufficient for RSP for a broad class of quantum channels. However, recent results show the property is not universal.
- One must aware of the fact that discord could be increased by LOCC.

Understanding Discord.....

- Streltsov and Zurek, Quantum discord cannot be shared, PRL, 111, 040401(2013); Zwolak and Zurek, Complementarity of quantum discord and classically accessible information, Sci Rep. 2013; 3: 1729.
- If a state of a composite system can be assembled by LOCC only then we call such states as separable, i.e., states which are not entangled.
- Now consider the situation where one attempts to pull apart a quantum state so that all the ingredients are classical and can be communicated classically to

- distant recipients. The cost of such operation is actually given by quantum discord.
- Thus one could consider discord as the information lost when a composite quantum state is disassembled.



- Initially the system S is correlated with the apparatus A. The recipient R is not correlated with SA. After LOCC we want the full information about the system S present in A should be transferred to R.
- Clearly, the recipient can obtain full information by LOCC about the system S is its classical information. The information which could not be transferred is thus quantum. So any state with non-zero quantum discord contains non-classical information.
- In the other work of Zurek, they showed an anti-symmetry property relating accessible information and discord.

Criteria for measures of correlation

- To formalize the new paradigm beyond entanglement one could set some properties for a measure of correlation.
- Modi et al., provided a set of conditions for a measure of correlation.
- Necessary conditions:
- Product states have no correlation
- Invariance under local unitarity
- Non=negativeness
- Classical states do not have quantum correlation(!)

Other conditions

- Reasonable conditions:
- Continuity under small perturbations
- Other type of strong and weak continuities
- Questionable/debatable conditions:
- For pure bipartite states total, classical and quantum correlations could be defined by nthe marginals
- Additivity total= classsical+quantum
- Classical and/or quantum are nonincreasing under LOCC
- Symmetry under interchange of subsystems

Some other important measures of correlation

- Quantum deficit,
- measurement induced disturbance,
- quantum dissonance,
- quantum dissension,
- measurement induced non-locality,
- local quantum uncertainty,
- etc….

Measurement Induced Nonlocality

- Consider the state, $\rho = \frac{1}{2}\left[|00\rangle\langle 00| + |11\rangle\langle 11|\right]$
- The state has non-zero value of a new measure of correlation is the Measurement Induced Non-Locality(MIN) .
- It is defined as,

$$N(\rho) = \max \|\rho - \Pi(\rho)\|$$

where the maximum is taken over all Von-Neumann measurements that preserves density matrix of the first party.

- Physically, MIN quantifies the global effect caused by locally invariant measurement.
- MIN vanishes for product state and remains positive for entangled states. For pure bipartite state MIN reduces to linear entropy like geometric discord.
- It has explicit formula for $2\otimes m$ system, $m\otimes n$ system(if reduced density matrix of first party is non-degenerate) system.

- MIN is invariant under local unitary.
- The set of states with zero MIN is a proper subset of the set of states with zero Discord. Thus, it signifies the existence of non-locality without Discord. The set of all zero MIN states is also non-convex.

Local Quantum Uncertainty

- Consider the Bell state **(1/√2)(|00> +|11>)**
- This state is an eigen-state of the global spin observable along *z z direction. H*ence measurement of this observable on the state is certain.
- However it can't be an eigen-state of any local observable in arbitrary spin direction **a.*I*,** and hence the measurement is inherently uncertain. In fact this is true for any pure entangled state and uncorrelated states such as|00> admits at least one certain observable.

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- **How to measure this uncertainty?**
- **How to extend this idea to mixed state also?**
- For this we need to build a measure and which does not affected by *classical mixing.*
- Classically, it is possible to measure any two observables with arbitrary accuracy. However, such measurement is not always possible in quantum systems. Uncertainty relation provides the statistical nature of errors in these kind of measurements. Measurement of single observable can also help to detect uncertainty of a quantum observable.

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- For a quantum state, an observable is called *quantum certain if the* error in measurement of the observable is due to only the ignorance about the classical mixing in that state. A good quantifier of this uncertainty is the skew information.
- For a bipartite quantum state *AB, Girolami et.al. (PRL,*110, 240402 (2013)) introduced the concept of local quantum uncertainty(LQU) and it is defined as the minimum over all local *maximally informative observable (or non-degenerate spectrum) of skew information for the state. This quantity* quantifies the minimum amount of uncertainty in a quantum state.

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- Non-zero value of this quantity indicates the non existence of any quantum certain observable for the state *AB*.
- It vanishes for all zero discord state w.r.t. measurement on party A.
- It is invariant under local unitary.
- It reduces to entanglement monotone for pure state. In fact, for pure bipartite states it reduces to linear entropy of reduced subsystems.
- So, LQU can be taken as a measure of bipartite quantumness.

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- Geometrically, LQU in a state *of a 2 × n system is the minimum* Hellinger distance between *and the state after a least disturbing* root-of-unity local unitary operation applied on the qubit.

Thanks to all present.