Noncritical-Topological Correspondence: Disc Amplitudes and Noncompact Branes

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Motivation

- There is one class of string backgrounds for which an exact nonperturbative solution is available.
- These are the noncritical type 0 strings. They have a Liouville formulation as well as a random matrix formulation.
- The latter has been used to derive, for example, the nonperturbative free energy – which we will study in this talk. This can probably be extended to correlation functions.
- This seems like the best place to test basic properties of string theory including for example string field theory, open-closed string duality and flux backgrounds.

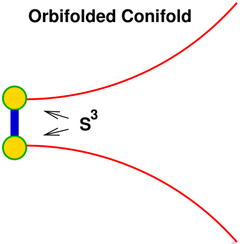
- Since noncritical strings are dual to topological strings, this provides an opportunity to understand the latter using the powerful techniques of random matrices.
- This in turn can help us understand superstring theory (of which topological strings are a sector).
- In this talk we will study some issues related to disc amplitudes and fluxes in type 0A string theory, and their counterparts in topological string theory..

Outline

Introduction

- It has been known for some time that noncritical string theory in two spacetime dimensions corresponds to a critical ($c_{top} = 9$) topological string *(Witten 1992, Mukhi-Vafa 1993)*.
- Analysis of the ground ring of the $c = 1$ string at self-dual radius *(Witten 1991, Ghoshal: Ph.D. thesis 1993)* led to the prediction that the dual topological string lives on a deformed conifold *(Ghoshal-Vafa 1995)*.
- Because the bosonic $c = 1$ string is nonperturbatively unstable, it has not been possible to extend this equivalence to the nonperturbative level.

- Such an extension can be explored in the case of the nonperturbatively stable Type 0 string theories in two dimensions.
- For these too, a description has been proposed in terms of topological string theory *(Ita et al 2004, Danielsson et al 2004, Hyun et al 2005)*.
- The Calabi-Yau dual to noncritical type 0A strings at a special radius ($R = 1$ in suitable units) was proposed to be a $\mathbb{Z}_2$ orbifold of the conifold:



- One of the most interesting aspects of noncritical type-0 strings is the possibility of turning on background Ramond-Ramond fluxes *(Takayanagi-Toumbas 2003, Douglas et al 2003, Gukov et al 2003, Maldacena-Seiberg 2005)*.
- In what follows, we discuss the role of these fluxes in the noncritical-topological correspondence, with emphasis on effects related to D-branes and nonperturbative contributions.
- Our main result is an improved correspondence in which the topological string reproduces the exact nonperturbative partition function of the type 0A string.
- This includes some subtle flux-dependent terms discovered recently by Maldacena and Seiberg.

Outline

Noncritical-topological duality

- Let us review some relevant aspects of topological string theory on noncompact Calabi-Yau spaces.
- The simplest example is the deformed conifold, described by the equation

$$zw - px = \mu$$

where z, w, p, x are complex coordinates of $\mathbb{C}^4$.
- Here μ is complex and its modulus determines the size of the S^3 that deforms the conifold.
- The topological B model is a theory of quantised deformations of the complex structure of the Calabi-Yau.

- The original noncritical-topological duality Ghoshal and Vafa says that if μ_M is the cosmological constant of the noncritical string:

$$S_{Liouville} \rightarrow S_{Liouville} + \mu_M \int d^2\xi \; e^{2\phi}$$

then the noncritical string is dual to the topological B-model with:

$$\mu = i g_s \mu_M$$

- This requires the genus-g partition functions of the two theories to coincide. Let:

$$\mathcal{F}^{c=1}(\mu_M) = \sum_{g=0}^{\infty} \mathcal{F}_g^{c=1}(-1)^{g-1}(g_s\mu_M)^{2-2g}$$

$$\mathcal{F}^{top,DC}(\mu) = \sum \mathcal{F}_g^{top,DC} \mu^{2-2g}$$

- The claim then amounts to:

$$\mathcal{F}_g^{c=1} = \mathcal{F}_g^{top,DC}, \quad \text{all } g$$

for which ample evidence has been found *(Antoniadis et al 1995, Morales et al 1997)*.

- There is also expected to be a 1-1 correspondence between the physical observables (tachyons in the $c = 1$ case and deformations of S^3 in the B-model case) and their correlators (e.g. "S^3 cosmology" of *Gukov-Saraikin-Vafa 2005*).

- For integer multiples of the self-dual radius, the corresponding topological theory lives on a $\mathbb{Z}_n$ orbifold of the conifold geometry. This too follows from the ground ring *(Ghoshal-Jatkar-Mukhi 1992)*.
- The deformed version of this space is described by the equation:

$$zw - \prod_{k=1}^{n}(px - \mu_k) = 0$$

which has n homology 3-spheres of size $\mu_1, \mu_2, \ldots, \mu_n$, each concealing one of the singularities.
- The geometry develops a conifold singularity if any of the μ_i's become zero, and a line singularity if $\mu_i = \mu_j$ for $i \neq j$.

- Consider the noncritical string at radius $R = n$ with only the cosmological perturbation μ_M turned on.
- The n parameters of the deformed conifold must be determined in terms of μ_M. In fact it was shown that:

$$\mu_k = i g_s \frac{\mu_M + ik}{n}, \quad k = -\tfrac{n-1}{2}, -\tfrac{n-1}{2} + 1, \cdots, \tfrac{n-1}{2}$$

(Gopakumar-Vafa 1998).

- These authors also argued that the free energy factorises into a sum of contributions:

$$\mathcal{F}^{R=n}_{c=1}(\mu) = \mathcal{F}^{top,DOC_n}(\{\mu_k\}) = \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \mathcal{F}^{top,DC}(\mu_k)$$

In type 0A strings we will find a nonperturbatively exact version of this result.

- This factorisation, among other things, can be understood in the Riemann surface formulation *(Aganagic et al 2003)*.
- In this approach one thinks of the following class of noncompact Calabi-Yaus:

$$zw - H(p, x) = 0$$

as a fibration described by the pair of equations:

$$zw = H, \quad H(p, x) = H$$

The fibre is $zw = H$, a complex hyperbola, and the base is the complexified p, x plane.

- Above points in the base satisfying $H(p, x) = 0$, the fibre degenerates to $zw = 0$, a pair of complex planes intersecting at the origin.
- Such points in the base form a Riemann surface which governs the physics of the topological string theory. The function $H(p, x)$ plays the role of a Hamiltonian and is related to integrability.

- For the orbifolded conifold, the Hamiltonian is:

$$H(p,x) = \prod_{k=1}^{n} (px - \mu_k)$$

so the Riemann surface $H(p,x) = 0$ factorises into disjoint branches.

- This is the physical reason for the factorisation of the free energy into a sum of contributions, one for each branch of the Riemann surface.
- As we already said, the above statements are meaningful only at the level of string perturbation theory.

Outline

Type 0 noncritical strings

- The type 0A string has, besides the cosmological constant μ_M, two additional (quantised) parameters q and $\tilde{q}$ *(Takayanagi-Toumbas 2003, Douglas et al 2003, Gukov et al 2003, Maldacena-Seiberg 2005)*.
- In the Liouville description these are the fluxes of two distinct Ramond-Ramond 2-form field strengths, $F_{t\phi}, \tilde{F}_{t\phi}$.
- The theory has a symmetry, labelled S-duality:

$$\mu_M \to -\mu_M, \quad F \leftrightarrow \tilde{F}$$

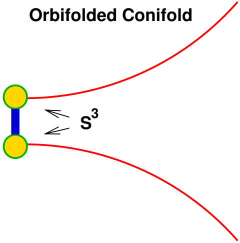
- In the matrix quantum mechanics (MQM) description, the fluxes have an asymmetric origin. For $\mu_M < 0$:

$$q = \#(D0) - \#(\overline{D0}), \qquad \tilde{q} = \text{Chern-Simons coefficient}$$

and the opposite for $\mu_M > 0$.

Special radius

- The Euclidean type 0A theory has a special value of the radius, $R = 1$ (in units where $\alpha' = 2$).
- At this radius the topological correspondence is simplest. The dual geometry is a $\mathbb{Z}_2$ orbifold of the conifold:



$$\begin{aligned} \mu &= \quad ig_s(\mu_M - \tfrac{i\hat{q}}{2}) &&= \frac{g_s}{2} y \\ \mu' &= -ig_s(\mu_M + \tfrac{i\hat{q}}{2}) &&= \frac{g_s}{2} \bar{y} \end{aligned}$$

where $y = \hat{q} + 2i\mu_M, \ \hat{q} = |q| + |\tilde{q}|$.

- The equation of the deformed, orbifolded conifold is:

$$zw + (px - \mu)(px - \mu') = 0$$

- Notice that complex conjugation exchanges the moduli of the two S^3's and acts as S-duality of the noncritical string.

- Following *Hyun et al*, it is convenient to use a duality that takes us to the topological B-model on a resolved orbifolded conifold with D-branes *(Gopakumar-Vafa 1998)*.
- On this space there are two 2-spheres (P^1's) which have respectively N_1, N_2 2-dimensional B-branes wrapped over them, where:

$$
\begin{aligned}
N_1 &= \frac{y}{2} = \frac{\hat{q}}{2} + i\mu_M \\
N_2 &= \frac{\bar{y}}{2} = \frac{\hat{q}}{2} - i\mu_M
\end{aligned}
$$

- The number of branes in this correspondence is complex, though remarkably $N_1 + N_2 = \hat{q}$ is real and integer.

- Now we have:

$$\mathcal{F}^{top,DOC}(\mu, \mu') = \mathcal{F}^{top,ROC}\left(N_1 = \frac{y}{2}, N_2 = \frac{\bar{y}}{2}\right)$$
$$= \mathcal{F}^{top,RC}\left(N = \frac{y}{2}\right) + \mathcal{F}^{top,RC}\left(N = \frac{\bar{y}}{2}\right)$$

- Here, factorisation of the B-brane contributions has been assumed. This will be justified later.
- On an ordinary resolved conifold, the free energy of N D-branes is given by the log of the matrix integral:

$$e^{-\mathcal{F}^{top,RC}(N)} = \frac{1}{\mathrm{vol}(U(N))}\int dM e^{-\frac{1}{2}\mathrm{tr}M^2} = \frac{(2\pi)^{\frac{N^2}{2}}}{\mathrm{vol}(U(N))}$$

- Next we use *(Ooguri-Vafa 2002)*:

$$\mathrm{vol}(U(N)) = \frac{(2\pi)^{\frac{1}{2}(N^2+N)}}{G_2(N+1)}$$

where $G_2(x)$ is the Barnes double-Γ function defined by:

$$G_2(z+1) = \Gamma(z) G_2(z), \quad G_2(1) = 1$$

- Thus we find

$$-\mathcal{F}^{top,RC}\left(N = \frac{y}{2}\right) - \mathcal{F}^{top,RC}\left(N = \frac{\bar{y}}{2}\right)$$
$$= \left(\log G_2\left(\frac{y}{2}+1\right) - \frac{y}{4}\log 2\pi\right) + c.c.$$

- Let us compare the above with what we know about the noncritical string starting from the matrix model.
- A complete nonperturbative solution for the free energy of Type 0A noncritical strings at arbitrary radius R is given by *(Maldacena-Seiberg 2005)*:

$$-\mathcal{F}_{0A}(\mu_M, q, \tilde{q}, R) = \Omega(y, R) + \Omega(\bar{y}, R) + \frac{\pi \mu_M R}{2}(|q| - |\tilde{q}|)$$

where:

$$\begin{aligned}\Omega(y, R) \equiv & -\int_0^\infty \frac{dt}{t}\Big[\frac{e^{-\frac{yt}{2}}}{4\sinh\frac{t}{2}\sinh\frac{t}{2R}} - \frac{R}{t^2} + \frac{Ry}{2t} \\ & + \left(\frac{1}{24}\left(R + \frac{1}{R}\right) - \frac{Ry^2}{8}\right)e^{-t}\Big]\end{aligned}$$

The integral is convergent for $\mathrm{Re}\, y > -\left(1 + \frac{1}{R}\right)$.

- At the special radius $R = 1$ it is easily shown from the integral form that:

$$\Omega(y, R = 1) = \log G_2\left(\frac{y}{2} + 1\right) - \frac{y}{4}\log 2\pi$$

where G_2 is the Barnes function discussed above.
- If we temporarily ignore the last term, we see that the free energy obeys holomorphic factorisation.
- Moreover, each factor is the (complexified) free energy of the bosonic $c = 1$ string at radius R. This justifies our assumption that the two B-brane contributions factorise.

- Note that from the 0A point of view, holomorphic factorisation follows from the answer for the free energy.
- From the topological point of view, however, it is an assumption which must be true for the correspondence to hold.

Rational radius

- Continuing to ignore the last term, we provide a simple, general and nonperturbatively exact derivation that the free energy of type 0A at any rational radius factorises into unit radius contributions.
- Let $R = \frac{p}{p'}$, with p and p' co-prime.
- Inserting this into the integral representation for Ω, and defining:

$$y_{k,k'} = \frac{y - p' + (2k'-1)}{p'} + \frac{-p + (2k-1)}{p},$$
$$k = 1, 2, \ldots, p; \quad k' = 1, 2, \ldots, p'$$

one can show that:

$$\Omega\left(y, R = \frac{p}{p'}\right) = \sum_{k,k'} \Omega(y_{k,k'}, R = 1) - \left(\frac{1}{24}\left(\frac{p}{p'} + \frac{p'}{p}\right) - \frac{py^2}{8p'}\right) \log p'$$

- The proof relies only on manipulating convergent integral representations.
- We see that the free energy at rational radius factorises into $2pp'$ distinct contributions, of which pp' are holomorphic in y and the remaining are anti-holomorphic.
- Each of the contributions corresponds to a theory at $R = 1$, or equivalently to the contribution of topological B-branes.
- The factorisation is exact upto an analytic and therefore non-universal term.
- This strongly suggests that the noncritical-topological correspondence is nonperturbatively exact.

- Again, from the topological point of view it remains a puzzle why the correspondence is exact. It means the $2pp'$ different B-branes do not seem to communicate with each other, e.g. by open strings!
- Perhaps this is due to factorisation of the underlying Riemann surface as well as the topological nature of the theory.
- In the deformed orbifolded conifold picture, the total space is:

$$zw - \prod_{k,k'}(px - \mu_{k,k'}) \prod_{k,k'}(px - \bar{\mu}_{k,k'})$$

where

$$\mu_{k,k'} = \frac{g_s}{2} y_{k,k'}$$

and therefore the associated Hamiltonian is:

$$H(p,x) = \prod_{k,k'}(px - \mu_{k,k'}) \prod_{k,k'}(px - \bar{\mu}_{k,k'})$$

Disc amplitudes and noncompact branes

- We now return to the puzzle about the extra term in the free energy.
- The noncritical string depends on three parameters, $q, \tilde{q}, \mu_M$, which in the continuum Liouville description arise as the two independent RR fluxes and the cosmological constant.
- However, the topological dual only depends on the complex number $y = |q| + |\tilde{q}| + 2i\mu_M$, and therefore on only two of these three parameters.
- It reproduces most of the free energy, which indeed depends only on two parameters and is the sum of mutually complex conjugate terms.

- But as we saw, the extra term in the free energy:

$$\mathcal{F}^{disc,2} = -\frac{\pi R}{2}\mu_M(|q| - |\tilde{q}|)$$

remains unaccounted for.

- This term is responsible for an important effect *(Maldacena-Seiberg 2005)*:
- From the factorised part of the free energy, the following disc contribution arises in the limit of large μ_M and fixed $\hat{q}$:

$$\mathcal{F}^{disc,1} = +\frac{\pi R}{2}|\mu_M|(|q| + |\tilde{q}|)$$

- Hence the total disc amplitude is:

$$\mathcal{F}^{disc} = \mathcal{F}^{disc,1} + \mathcal{F}^{disc,2} = \frac{\pi R}{2}\Big[(|\mu_M| - \mu_M)|q| + (|\mu_M| + \mu_M)|\tilde{q}|\Big]$$

- This can be written as:

$$\begin{aligned}\mathcal{F}^{disc} &= (2\pi R)\frac{\mu_M}{2}|\tilde{q}|, \quad \mu_M > 0 \\ &= (2\pi R)\frac{|\mu_M|}{2}|q|, \quad \mu_M < 0\end{aligned}$$

- The physical interpretation is that for $\mu_M > 0$ the RR flux of $\tilde{q}$ units associated to the gauge field $\tilde{A}$ is supported by $|\tilde{q}|$ ZZ branes in the vacuum.
- The contribution per brane to the free energy is given by the product of its extent in Euclidean time $2\pi R$ and its tension $\frac{|\mu_M|}{2}$. The other flux of q units associated to the gauge field A has no source.
- For $\mu_M < 0$ the situation is reversed.

- In the absence of the term $\mathcal{F}^{disc,2}$ there is no satisfactory physical interpretation of the disc amplitude in terms of ZZ branes. This makes the term extremely important for a consistent noncritical string theory.
- We now propose that the missing term is supplied, on the topological side, by additional topological branes.
- These are noncompact B-branes wrapping a degenerate fibre of the Calabi-Yau over the Riemann surface $H(p, x) = 0$.

- First consider the case $R = 1$ for which the Riemann surface has two branches.
- Place a single noncompact B-brane along one branch of the degenerate fibre over a point x on the Riemann surface.
- The brane is asymptotically at x_* but its interior region has been moved to x. Such branes are BPS.
- The action of such a brane is a reduction of holomorphic Chern-Simons theory and has been shown to be *(Aganagic-Vafa 2000, Aganagic et al 2003)*:

$$S(x) = \frac{1}{g_s} \int_{x_*}^{x} p(z)\, dz$$

- For the case of interest to us, the Riemann surface consists of two disjoint factors:

$$xp = \frac{g_s}{2} y, \quad xp = \frac{g_s}{2} \bar{y}$$

Thus a brane on the first branch contributes:

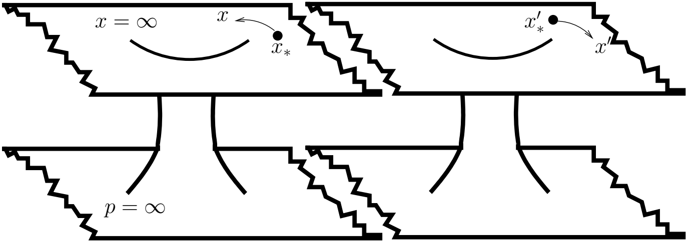
$$S(x) = \frac{\mu}{g_s} \ln \frac{x}{x_*}$$

- Let us now place one noncompact brane above each of the two branches, and take their asymptotic positions to be at x_*, x'_* which will both be sent to infinity.
- Then their total contribution to the free energy is:

$$S(x, x') = \frac{1}{2} \left(y \ln \frac{x}{x_*} + \bar{y} \ln \frac{x'}{x'_*} \right)$$

- Now we will choose our branes such that x, x' are also at infinity, but rotated by angles θ, θ' respectively along the circle at infinity relative to the original points x_*, x'_*:

$$x = x_*\, e^{i\theta}, \quad x' = x'_*\, e^{i\theta'}$$



- It follows that:

$$\begin{aligned} S(x_1, x_2) &= \frac{i}{2}(y\,\theta + \bar{y}\,\theta') \\ &= -\mu_M(\theta - \theta') + i\frac{\hat{q}}{2}(\theta + \theta') \end{aligned}$$

- The factors of g_s have conveniently cancelled out, and the real part of the above contribution is proportional to μ_M.
- Now if we choose:

$$\theta = -\theta' = \frac{\pi}{4}(|q| - |\tilde{q}|)$$

we find that the noncompact branes give a contribution:

$$S = -\frac{\pi}{2}\mu_M(|q| - |\tilde{q}|)$$

to the free energy, precisely equal to the desired disc term at $R = 1$.

- These considerations can be extended to other rational radii.
- For a radius $R = \frac{p}{p'}$, we have $2pp'$ branches for the Riemann surface. So noncompact branes can be placed with:

$$x_{k,k'} = x_{*\,k,k'}\, e^{i\theta_{k,k'}}$$

on the first pp' asymptotic regions, and the opposite phases on the remaining pp' regions. Here:

$$\theta_{k,k'} = \frac{\pi}{4p'^2}\left(|q| - |\tilde{q}|\right), \quad \text{all } k, k'$$

- The net contribution of these to the free energy is then:

$$-2\mu_M \sum_{k=1}^{p} \sum_{k'=1}^{p'} \theta_{k,k'} = -\frac{\pi p}{2p'}\, \mu_M(|q| - |\tilde{q}|)$$

in accordance with the extra disc term for $R = \frac{p}{p'}$.

- It is quite nontrivial that we were able to reproduce the subtle disc term by a simple configuration of noncompact branes in every case.
- The $\frac{1}{g_s}$ factor in front of the holomorphic Chern-Simons action, and the g_s in the complex-structure moduli

$$\mu_{k,k'} = \frac{g_s}{2}\left(\frac{\hat{q}+2i\mu_M-p'+(2k'-1)}{p'} + \frac{-p+(2k-1)}{p}\right)$$

  exactly cancel out.
- Moreover, $\mu_{k,k'}$ all have a common imaginary part proportional to μ_M. These facts were important in allowing us to obtain the desired contribution from noncompact branes.

Outline

Conclusions

- One of our main results has been that the noncritical-topological correspondence for type 0 noncritical strings has to include noncompact branes on the topological side.
- This introduces a dependence on a new parameter which we interpret as $|q| - |\tilde{q}|$ on the noncritical side, and renders the duality consistent with the dependence of the noncritical theory on three parameters: μ_M, q and $\tilde{q}$.

- For the future:
  (i) understand the dictionary more precisely. What are ZZ branes on the topological side?
  (ii) the noncritical side is physically inconsistent without the subtle disc term. Is the topological side inconsistent without noncompact branes?
  (iii) what is the origin of exact factorisation on the topological side?
  (iv) generalise 0A exact solution to correlators and a nonperturbatively defined Normal Matrix Model.
  (v) topological-anti-topological point of view.