Black Hole Entropy Function, Attractors and Precision State Counting

1. A brief introduction

2. Black hole entropy function

3. Application to $\mathcal{N} = 4$ SUSY theories

4. Statistical entropy of $\mathcal{N} = 4$ SUSY theories

References: Many

Other directly related talks at this meeting

Dabholkar, Astefanesei, Jatkar, Sahoo, Jena, Goldstein, Mahapatra

List of collaborators

Entropy Function:

D. Astefanesei, A. Dabholkar, K. Goldstein, N. Iizuka, A. Iqubal, R. P. Jena, B. Sahoo, M. Shigemori, S. P. Trivedi

Dyon state counting

J. R. David, D. P. Jatkar

String theory gives a framework for studying classical and quantum properties of black holes.

One of the important properties characterizing a black hole is the Bekenstein-Hawking entropy S_{BH}.

For a two derivative action

$$S_{BH} = A/(4G_N)$$

A: Area of the event horizon

G_N: Newton's constant

Question: Can we understand this entropy from statistical viewpoint?

In string theory one finds that for a wide class of extremal black holes

$$S_{BH} = S_{stat}, \qquad S_{stat} \equiv \ln(Degeneracy)$$

Given this success, it is natural to carry out our study of black holes to finer details.

String theory leads to

(Super-)gravity + higher derivative terms

Does the agreement continue to hold even after taking into account the effects of higher derivative corrections?

In order to attack this problem we need to open two fronts.

First of all we need to learn how to take into account the effect of the higher derivative terms on the computation of black hole entropy.

But we also need to know how to calculate the statistical entropy to greater accuracy.

$\rightarrow$ involves precise computation of the degeneracy of states with a given set of charges.

In this talk we shall address both problems.

A general frameork for computing higher derivative corrections to black hole entropy has been developed by Wald.

We shall use this to develop a general method for calculating higher derivative corrections to S_{BH} for **extremal** black holes.

We shall also calculate the exact degeneracy of microstates associated with a certain set of black holes in string theory.

Finally we compare the black hole entropy with the statistical entropy computed from microscopic counting of states.

How do we define extremal black holes in a higher derivative theory?

Consider spherically symmetric black holes in $D=4$.

All known extremal black holes in $D=4$ have near horizon geometry $AdS_2 \times S^2$ with isometry $SO(2,1)\times SO(3)$.

We shall take this to be the definition of extremal black holes even in the presence of higher derivative terms.

Consider an arbitrary general coordinate invariant theory of gravity coupled to a set of Maxwell fields $A_\mu^{(i)}$ and neutral scalar fields $\{\phi_s\}$.

The most general form of the near horizon geometry of an extremal black hole consistent with $SO(2,1) \times SO(3)$ isometry:

$$\begin{aligned} ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu &= v_1 \left(-r^2 dt^2 + \frac{dr^2}{r^2} \right) \\ &\quad + v_2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \\ \phi_s &= u_s \end{aligned}$$

$$F_{rt}^{(i)} = e_i, \qquad F_{\theta\phi}^{(i)} = \frac{p_i}{4\pi} \sin\theta \,,$$

v_1, v_2, u_s, e_i, p_i: constants

$$ds^2 = v_1 \left(-r^2 dt^2 + \frac{dr^2}{r^2} \right) + v_2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right)$$
$$\phi_s = u_s$$
$$F^{(i)}_{rt} = e_i, \qquad F^{(i)}_{\theta\phi} = \frac{p_i}{4\pi} \sin\theta \,,$$

$\sqrt{-\det g}\,\mathcal{L}$: Lagrangian density.

Take $\mathcal{L}$ to be invariant under general coordinate, local Lorentz and gauge transformations.

Define:

$$f(\vec{u}, \vec{v}, \vec{e}, \vec{p}) \equiv \int d\theta \, d\phi \, \sqrt{-\det g}\, \mathcal{L}$$

$$\mathcal{E}(\vec{u}, \vec{v}, \vec{e}, \vec{q}, \vec{p}) \equiv 2\pi (e_i \, q_i - f(\vec{u}, \vec{v}, \vec{e}, \vec{p}))$$

Using equations of motion and Wald's formula for entropy one finds that for an extremal black hole of electric charge $\vec{q}$ and magnetic charge $\vec{p}$,

1. the values of $\{u_s\}$, $\{e_i\}$, v_1 and v_2 are obtained by extremizing $\mathcal{E}(\vec{u}, \vec{v}, \vec{e}, \vec{q}, \vec{p})$ with respect to $\{u_s\}$, $\{e_i\}$, v_1 and v_2:

$$\frac{\partial \mathcal{E}}{\partial u_s} = 0, \quad \frac{\partial \mathcal{E}}{\partial v_1} = 0\,, \quad \frac{\partial \mathcal{E}}{\partial v_2} = 0, \quad \frac{\partial \mathcal{E}}{\partial e_i} = 0$$

2. $S_{BH} = \mathcal{E}$ at the extremum.

Thus the 'entropy function' $\mathcal{E}$ determines

– the near horizon values $\{u_s\}$ of the scalar fields,

– the sizes v_1, v_2 of AdS_2 and S^2

– the gauge field strengths $\{e_i\}$

– the entropy S_{BH}

The results can be generalized to

1. Rotating black holes

2. Black holes in higher dimensions

3. Lagrangian densities containing Chern-Simons terms.

The entropy function formalism leads to a general proof of the 'attractor mechanism' for extremal black holes.

If we have a theory with scalar fields which have no potential then asymptotically the scalar fields can have arbitrary values.

However the entropy of an extremal black hole with a given set of charges $\vec{q}$, $\vec{p}$ is independent of the asymptotic values of these 'moduli fields'.

Proof of attractor mechanism:

If $\mathcal{E}$ has no flat directions then the extremization of $\mathcal{E}$ determines the near horizon parameters $\vec{u}$, $\vec{v}$, $\vec{e}$ completely in terms of $\vec{q}$, $\vec{p}$.

If $\mathcal{E}$ has flat directions, then extremization of $\mathcal{E}$ does not determine $\vec{u}$, $\vec{v}$, $\vec{e}$ uniquely and there is a continuous family of extrema.

But since $\mathcal{E}$ does not depend on the flat directions, $S_{BH} = \mathcal{E}$ is still determined in terms of $\vec{q}$, $\vec{p}$ and is independent of any other data.

The entropy function formalism has been successfully used for calculating higher derivative corrections to the entropy of extremal black holes in many theories.

We shall now apply this to compute the entropy of a special class of quarter BPS black holes in a class of $\mathcal{N} = 4$ supersymmetric string theories.

CHL models based on $\mathbb{Z}_N$ orbifolds

1. Begin with heterotic string theory on

$$T^4 \times S^1 \times \widehat{S}^1$$

   T^4: A four torus

   S^1, $\widehat{S}^1$: two circles with period 2π

2. Take the orbifold by a $\mathbb{Z}_N$ group generated by

   $2\pi/N$ shift along S^1 together with an order N internal symmetry of heterotic string theory on T^4.

A dual description

1. Begin with type IIB string theory on

$$K3 \times S^1 \times \widetilde{S}^1$$

2. Take the orbifold by a $\mathbb{Z}_N$ group generated by $2\pi/N$ shift along S^1 together with an appropriate order N internal symmetry of type IIB string theory on $K3$.

The resulting theory is $\mathcal{N} = 4$ supersymmetric.

A special class of values of N:

$$N = 1, 2, 3, 5, 7$$

$N = 1$: heterotic string theory on T^6.

For these theories the rank of the gauge group is

$$r = 2k + 8, \qquad k = \frac{24}{N+1} - 2$$

At a generic point in the modul space we have

1. The string metric $G_{\mu\nu}$.

2. r U(1) gauge fields $A_\mu^{(i)}$

3. $r \times r$ matrix valued scalar field M satisfying

$$M^T L M = L, \qquad M^T = M,$$

L: A matrix with six eigenvalues $+1$ and $r-6$ eigenvalues -1.

4. Dilaton-axion field (S, a)

Near horizon field configuration:

$$ds^2 = v_1 \left(-r^2 dt^2 + \frac{dr^2}{r^2} \right) + v_2 (d\theta^2 + \sin^2\theta \, d\phi^2) \,,$$

$$S = u_S, \qquad a = u_a, \qquad M_{ij} = u_{Mij}$$

$$F^{(i)}_{rt} = e_i, \quad F^{(i)}_{\theta\phi} = \frac{p_i \sin\theta}{4\pi}, \quad i, j = 1, \ldots r$$

First consider the case where we keep only two derivative terms in the action.

1. Compute the entropy function $\mathcal{E}$ using

$$\mathcal{E} \equiv 2\pi \left(e_i q_i - \int d\theta \, d\phi \, \sqrt{-\det g} \, \mathcal{L} \right)$$

2. Eliminate the variables e_i, u_M, v_1, v_2 by extremizing $\mathcal{E}$ with respect to them.

Result for $P^2 Q^2 > (P \cdot Q)^2$:

$$\mathcal{E} = \frac{\pi}{2}\left[\left(\frac{Q^2}{u_S} + \frac{P^2}{u_S}(u_S^2 + u_a^2) - 2\,\frac{u_a}{u_S}\,Q \cdot P\right)\right]$$

$$Q_i \equiv 2q_i, \qquad P_i \equiv \frac{1}{4\pi}\,L_{ij}p_j$$

$$P^2 \equiv P^T L P, \quad Q^2 \equiv Q^T L Q, \quad Q \cdot P \equiv Q^T L P$$

Eliminate u_a, u_S:

$$S_{BH} = \mathcal{E} = \pi\sqrt{Q^2 P^2 - (Q \cdot P)^2}$$

This gives the supergravity result for black hole entropy.

Now consider the effect of a special type of higher derivative correction to the Lagrangian density:

$$\sqrt{-\det G}\Delta\mathcal{L}$$

$$= \phi_k(a,S)\sqrt{-\det G}\left\{R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right\}$$

$\phi_k(a,S)$: an S-duality invariant function.

For the $\mathbb{Z}_N$ CHL models with prime N:

$$\begin{aligned}
\phi_k(a,S) &= -\frac{1}{64\pi^2}\Big((k+2)\ln S \\
&\quad + \ln f^{(k)}(a+iS) + \ln f^{(k)}(-a+iS)\Big) \\
f^{(k)}(\tau) &= \eta(\tau)^{k+2}\,\eta(N\tau)^{k+2} \\
k &= \frac{24}{N+1} - 2
\end{aligned}$$

$\eta(\tau)$: Dedekind η-function

The addition of Gauss-Bonnet term in the action induces the following change in $\mathcal{E}$:

$$\Delta\mathcal{E} = 64\,\pi^2\,\phi_k(u_a, u_S)$$

Elimination of e_i, u_M, v_1, v_2 can be done as before since the extra term does not depend on these variables.

$$\begin{aligned}\mathcal{E}+\Delta\mathcal{E} \;&=\; \frac{\pi}{2}\Big[\left(\frac{Q^2}{u_S}+\frac{P^2}{u_S}(u_S^2+u_a^2)-2\,\frac{u_a}{u_S}\,Q\cdot P\right)\\ &\quad +128\,\pi\,\phi_k(u_a,u_S)\Big]\end{aligned}$$

Final result for entropy is obtained by eliminating u_a and u_S by extremizing $\mathcal{E}+\Delta\mathcal{E}$.

Computation of statistical entropy

Consider a generic 1/4 BPS dyonic state in CHL string theory carrying r dimensional electric charge vector Q and magnetic charge vector P.

What is the degeneracy $d(Q,P)$ of these states?

We shall derive the formula for $d(Q,P)$ for a specific class of charge vectors (Q,P).

Then we shall express the formula in terms of T-duality invariant combinations P^2, Q^2 and $Q \cdot P$.

We consider a configuration in type IIB string theory on $K3 \times S^1 \times \widetilde{S}^1/\mathbb{Z}_N$ with

1) Q_5 D5-branes wrapped on $K3 \times S^1$,

2) Q_1 D1-branes wrapped on S^1, and

3) one Kaluza-Klein monopole associated with $\widetilde{S}^1$ compactification

carrying $-n$ units of momentum along S^1 and J units of momentum along $\widetilde{S}^1$

T-duality invariants:

$$P^2 = 2Q_5(Q_1 - Q_5), \quad Q^2 = 2n/N, \quad Q \cdot P = J$$

In the weakly coupled type IIB description the low energy dynamics of the system is described by several non-interacting pieces:

1) The dynamics of the Kaluza-Klein monopole

2) The dynamics of the D1-D5 center of mass coordinate in the Kaluza-Klein monopole background

3) The relative motion between the D1 and the D5-brane

Each of these systems carry certain amount of momenta along S^1 and $\widetilde{S}^1$.

We calculate the 'partition function' associated with each of these three pieces and then take the product.

$\rightarrow$ degeneracy as a function of Q_1, Q_5, n, J.

Then we express the result in terms of P^2, Q^2 and $Q \cdot P$ using

$$P^2 = 2Q_5(Q_1 - Q_5), \quad Q^2 = 2n/N, \quad Q \cdot P = J$$

The result for $d(Q,P)$

$$d(Q,P) = \frac{1}{N}\int_C d\rho d\sigma dv \frac{1}{\Phi_k(\rho,\sigma,v)} \exp\left[-i\pi(\rho P^2 + \sigma Q^2 + 2vQ\cdot P)\right],$$

ρ, σ, v: complex parameters

The integration 'contour' C is defined to be the three real dimensional subspace:

$$Im\,\rho = M_1, \quad Im\,\sigma = M_2, \quad Im\,v = M_3,$$

$$0 \le Re\,\rho \le 1, \quad 0 \le Re\,\sigma \le N, \quad 0 \le Re\,v \le 1\,.$$

M_1, M_2, M_3: large real constants

Expression for Φ_k:

$$\Phi_k(\rho,\sigma,v) = \exp\left(2\pi i\left(\frac{1}{N}\sigma+\rho+v\right)\right)$$

$$\prod_{r=0}^{N-1} \prod_{\substack{l,b\in Z, k'\in Z+\frac{r}{N}\\ k',l,b>0}} \left\{1-\exp(2\pi i(k'\sigma+l\rho+bv))\right\}^{\sum_{s=0}^{N-1} e^{-2\pi i l s/N}\, c^{(r,s)}(4lk'-b^2)}$$

$k',l,b > 0$: $(k'>0, l\geq 0, b\in \mathbb{Z})$ or $(\ k'=0, l>0, b\in\mathbb{Z})$ or $(k'=0, l=0, b<0)$

$c^{r,s}(n)$: known coefficients, given in terms of jacobi ϑ-functions and Dedekind η-functions.

For comparing $\ln d(Q,P)$ to black hole entropy we need to estimate $d(Q,P)$ for large Q,P.

Strategy:

a) Do the v integral by picking up residues from the poles of $1/\Phi_k$

Result:

$$d(Q,P) = \int d\rho d\sigma e^{-F(\rho,\sigma)}$$

for some function $F(\rho,\sigma)$.

b) Then do the ρ and σ integral using saddle point approximation.

$$d(Q,P) = \int d\rho d\sigma e^{-F(\rho,\sigma)}$$

This integral can be regarded as a 'path integral' over zero dimensional fields ρ, σ with action $F(\rho,\sigma)$, and the result for $\ln d(Q,P)$ may be expressed as the result of extremizing an 'effecting action' $\Gamma(\rho,\sigma)$ with respect to ρ, σ.

$\ln d(Q,P)$ is the value of $-\Gamma(\rho,\sigma)$ at its extremum.

$-\Gamma(\rho,\sigma)$ can be calculated using Feynman diagrams and be called the statistical entropy function.

Result for Γ after a suitable change of variables from (ρ, σ) to (u_a, u_S):

$$
\begin{aligned}
-\Gamma(u_a, u_S) &= \frac{\pi}{2}\Big[\left(\frac{Q^2}{u_S} + \frac{P^2}{u_S}(u_S^2 + u_a^2) - 2\,\frac{u_a}{u_S}\,Q\cdot P\right) \\
&\quad +128\,\pi\,\phi_k(u_a, u_S)\Big] + \mathcal{O}(Q^{-2}, P^{-2})
\end{aligned}
$$

$$
\begin{aligned}
\phi_k(u_a, u_S) &= -\frac{1}{64\pi^2}\Big((k+2)\ln u_S \\
&\quad + \ln f^{(k)}(u_a + i u_S) + \ln f^{(k)}(-u_a + i u_S)
\end{aligned}
$$

$$
f^{(k)}(\tau) = \eta(\tau)^{k+2}\,\eta(N\tau)^{k+2}
$$

$$
k+2 = 24/(N+1)
$$

Now recall that the entropy function for the black hole, after extremization with respect to all the near horizon parameters except the values of the axion-dilaton field, is given by:

$$\mathcal{E} \;=\; \frac{\pi}{2}\Big[\left(\frac{Q^2}{u_S}+\frac{P^2}{u_S}(u_S^2+u_a^2)-2\,\frac{u_a}{u_S}\,Q\cdot P\right) \\ +128\,\pi\,\phi_k(u_a,u_S)\Big]$$

(u_a, u_S): near horizon value of the axion-dilaton field.

$\mathcal{E}$ and $-\Gamma$ are identical functions to this order.

Thus extremization of $\mathcal{E}$ and $-\Gamma$ give the same answer.

$\rightarrow$ equality between black hole entropy and statistical entropy to first non-leading power of inverse charges.

Thus we see that the formula for the statistical entropy matches the black hole entropy to this order.

This result can be generalized for

1. CHL models with non-prime values of N.

2. $\mathcal{N}=4$ supersymmetric $\mathbb{Z}_N$ orbifiolds of type IIA string theory on T^6.

An open question: What is the effect of other four derivative terms on the entropy?

When Q is large compared to P, u_S is large and tree level approximation is good in the heterotic description.

Effect of including the set of all tree level four derivative correction terms in the Lagrangian

$\rightarrow$ same as the one obtained by just using the Gauss-Bonnet term.

One can also give a general argument based on supersymmetry that tree level higher derivative terms do not modify the result.

When Q and P are of same order, then keeping only tree level terms is not a useful approximation scheme.

Thus we need to include the full $\phi_k(a, S)$ as coefficient of the Gauss-Bonnet term.

However there are other four derivative corrections to the effective action.

What is their effect on the entropy?

Is there a non-renormalization theorem similar to that for the tree level result?