Quantizing BPS Black Holes

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LPTHE and LPTENS, Paris

ISM 2006, Dec 13, India

Goal and Motivations

- Goal: perform a radial quantization of stationary, spherically symmetric, BPS solutions of $\mathcal{N} = 2, D = 4$ supergravity;
- Main motivation: evaluate (and improve on) OVV's holographic interpretation of the OSV conjecture

Ooguri Strominger Vafa; Ooguri Vafa Verlinde

- Second motivation: set up a general framework for constructing automorphic functions generating exact BH degeneracies as their Fourier coefficients, in the spirit of the DVV formula for $\mathcal{N} = 4$;
- Instill supersymmetry and holography into early discussions:

Breitenlohner Gibbons Maison (1988), Cavaglia de Alfaro Filippov (1995), Breitenlohner Hellmann (96)

- Work in collaboration with Günaydin, Neitzke, Waldron and more recently Rocek, Vandoren;

hep-th/0512296,hep-th/0607227, hep-th/0612xxx

BPS black holes in type II string theory compactified on CY_3 enjoy simplifying properties:

- By the attractor phenomenon, the near-horizon solution, hence the Bekenstein-Hawking entropy, depends only on the conserved charges;
- Being supersymmetric, they are expected to correspond to exact ground states of the quantum Hamiltonian at fixed charges, with an arbitrarily large degeneracy;
- The string coupling can be made arbitrary small throughout the geometry, allowing a description as a gas of weakly interacting open strings in the presence of D-branes.

Strominger Vafa; Johnson Khuri Myers; Maldacena Strominger Witten

$AdS_{3,2}/CFT_{2,1}$

- The modern understanding relies on AdS/CFT in the near horizon geometry $AdS_3 \times S^2 \times CY_3^*$. The central charge of the two-dimensional SCFT on the boundary, controlling the density of highly excited states via Cardy's formula, can be computed on geometrical grounds.

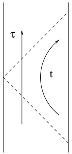
Brown Henneaux; Carlip; Strominger

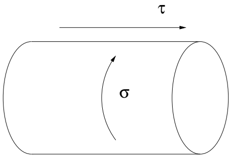
- AdS_3 is really the near horizon geometry of a 5D black string: if $[D6] \neq 0$ it is not possible to lift the 4D black hole to a black string in 5D. Moreover, such a lift may seem rather artificial as the M-theory direction can be made arbitrarily small.
- Instead, one may hope for a holographic description in terms of a superconformal quantum mechanics living at the boundary(ies) of AdS_2; no concrete proposal yet, except in some probe approximations.

$AdS_2/SCFT_1$ and channel duality

- A possible strategy is to indirectly compute the spectrum of the SQM via channel duality, as in open/closed string duality:

$$\mathrm{Tr} e^{-\pi t H_{open}} = \langle B | e^{-\frac{\pi}{t} H_{closed}} | B \rangle$$





Here, H_{closed} is the Hamiltonian for string theory in AdS_2 in radial quantization. The real interest is in H_{open}.

- This is hardly doable in general, but becomes tractable in a mini-superspace approximation, where one keeps only spherically symmetric SUGRA modes in the bulk. This approximation is hard to control, but perhaps justified in the BPS sector.

Topological amplitude and black hole wave function I

- Recently, OVV suggested that the OSV conjecture

$$\Omega(p^I, q_I) \sim \int d\phi^I \, |\Psi_{top}(p^I + i\phi^I)|^2 \, e^{\phi^I q_I}$$

can be interpreted just in this way,

$$\Omega(p, q) \sim \langle \Psi^+_{p,q} | \Psi^-_{p,q} \rangle$$

where

$$\Psi^{\pm}_{p,q}(\phi) = e^{\pm \frac{1}{2} q_I \phi^I} \, \Psi_{top}(p^I \mp i\phi^I)$$

- The main goal of this talk will be to perform a rigorous treatment of radial quantization, and evaluate / improve on OVV's proposal.

Outline

Outline

Stationary solutions and KK* reduction I

- Stationary solutions in 4D can be parameterized in the form

$$ds_4^2 = -e^{2U}(dt+\omega)^2 + e^{-2U} ds_3^2 \ , \quad A_4^I = \zeta^I dt + A_3^I$$

where $ds_3, U, \omega, A_3^I, \zeta^I$ and the 4D scalars $z^i \in \mathcal{M}_4$ are independent of time. In contrast to usual KK ansatz, the Killing vector is time-like.

- Such solutions can be described by reducing the D=3+1 action to three Euclidean dimensions. As usual, one-forms (A_3^I, ω) can be dualized into pseudo-scalars $(\tilde{\zeta}_I, a)$, where a is the twist (or NUT) potential.

Stationary solutions and KK* reduction II

- The result is 3D Euclidean gravity coupled to a non-linear sigma model on a pseudo-Riemannian space $\mathcal{M}_3^*$,

$$ds^2 = dU^2 + g_{ij}\, dz^i dz^j + e^{-4U}\left(da + \zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I\right)^2$$
$$-e^{-2U}\left[t_{IJ} d\zeta^I d\zeta^J + t^{IJ}\left(d\tilde{\zeta}_I + \theta_{IK} d\zeta^K\right)\left(d\tilde{\zeta}_J + \theta_{JL} d\zeta^L\right)\right]$$

  where g_{ij} is the metric on $\mathcal{M}_4$, and $\mathcal{N}_{IJ} := \theta_{IJ} - i t_{IJ}$ are the complexified gauge kinetic terms.

- $\mathcal{M}_3^*$ has a $2n_V + 3$-dimensional Heisenberg algebra of isometries

$$p^I = \partial_{\tilde{\zeta}^I} + \zeta_I\, \partial_a\,, \quad q_I = \partial_{\zeta^I} - \zeta_I\, \partial_a\,, \quad k = \partial_a$$
$$\left[p^I, q_J\right] = 2k\, \delta^I_J$$

Spherically symmetric BH and geodesics I

- Now, restrict to spherically symmetric solutions, with spatial slices

$$ds_3^2 = N^2(\rho) d\rho^2 + r^2(\rho) d\Omega_2^2$$

- The sigma-model action becomes, up to a total derivative,

$$S = \int d\rho \left[\frac{N}{2} + \frac{1}{2N} \left(\dot{r}^2 - r^2 g_{ab} \dot{\phi}^a \dot{\phi}^b \right) \right]$$

where g_{ab} is the metric on $\mathcal{M}_3^*$: this describes the (unparameterized) geodesic motion of a fiducial particle with unit mass on the cone $\mathbb{R}^+ \times \mathcal{M}_3^*$.

The Wheeler-DeWitt constraint I

- The equation of motion of N imposes the Hamiltonian constraint,or Wheeler-DeWitt equation

$$H_{WDW} = (p_r)^2 - \frac{1}{r^2} g^{ab} p_a p_b - 1 \equiv 0$$

- The gauge choice $N = r^2$ allows to separate the problem into radial motion along r, and affine geodesic motion on $\mathcal{M}_3^*$:

$$g^{ab} p_a p_b = C^2 \ , \quad (p_r)^2 - \frac{C^2}{r^2} - 1 \equiv 0 \quad \Rightarrow \quad r = \frac{C}{\sinh C\rho} \ ,$$

- $C = 2 T_H S_{BH}$ is the extremality parameter: extremal (in particular BPS) black holes correspond to light-like geodesics.

Isometries and conserved charges

- The conserved charges associated to the Heisenberg isometries correspond to the electric and magnetic charges (q_I, p^I) and the NUT charge k.
- If $k \neq 0$, the off-diagonal term in the 4D metric

$$ds_4^2 = -e^{2U}(dt + k\cos\theta d\phi)^2 + e^{-2U}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)]$$

  implies the existence of closed time-like curves around ϕ direction, near $\theta = 0$. Bona fide 4D black holes arise in the "classical limit" $k \to 0$. Keeping $k \neq 0$ will allow us to greatly extend the symmetry.
- The conserved charge associated to the extra isometry $\partial_U + \zeta^I\partial_{\zeta^I} + \tilde{\zeta}_I\partial_{\zeta_I} + 2\partial_a$ is the ADM mass; it does not commute with p, q, k.

Conserved charges and black hole potential

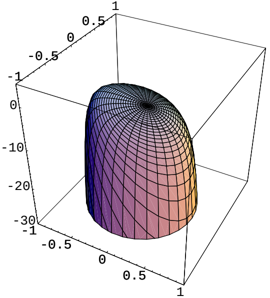
- Setting $k = 0$ for simplicity, one arrives at the Hamiltonian,

$$H = \frac{1}{2}\left[p_U^2 + p_i g^{ij} p_j - e^{2U} V_{BH}\right] \equiv C^2$$

where V_{BH} is the “black hole potential”,

$$V_{BH}(z^i, p^I, q_I) = \frac{1}{2}(q_I - \mathcal{N}_{IJ}p^J)t^{IK}(q_K - \bar{\mathcal{N}}_{KL}p^L) + \frac{1}{2}p^I t_{IJ} p^J$$

- The potential $V = -e^{2U} V_{BH}$ is unbounded from below.



Quantizing geodesic motion I

- The classical phase space is the cotangent bundle $T^*(\mathcal{M}_3^*)$, specifying the initial position and velocity: non compact.
- Quantization proceeds by replacing functions on phase space by operators acting on wave functions in $L_2(\mathcal{M}_3^*)$, subject to

$$\Delta_3 \Psi(U, z^i, \zeta^I, \tilde{\zeta}_I, a) = C^2 \Psi$$

  where Δ_3 is the Laplace-Beltrami operator on $\mathcal{M}_3^*$.
- The electric, magnetic and NUT charges may be diagonalized as

$$\Psi(U, z^i, \zeta^I, \tilde{\zeta}_I, a) = \Psi_{p,q}(U, z)\, e^{i(q_I \zeta^I + p^I \tilde{\zeta}_I)}$$

$$\left[-\partial_U^2 - \Delta_4 - e^{2U} V_{BH} - C^2 \right] \Psi_{p,q}(U, z) = 0$$

Quantizing geodesic motion II

- The black hole wave function $\Psi_{p,q}(U, z)$ describes quantum fluctuations of the 4D moduli as one reaches the horizon at $U \to -\infty$. Naively, should be peaked at the attractor point.
- Restoring the variable r, one could also describe the quantum fluctuations of the horizon area $r^2 e^{-2U}$, around the classical value $4S_{BH}(p, q)$.
- The natural inner product is the Klein-Gordon inner product at fixed U, famously NOT positive definite. A standard remedy in quantum cosmology is "third quantization", possibly relevant for black hole fragmentation / multi-centered solutions.

Outline

Attractor flow in $N = 2$ supergravity

- Consider $N = 2$ SUGRA coupled to n_V abelian vector multiplets: the vector multiplet scalars z^i take values in a special Kähler manifold $\mathcal{M}_4$. Hypers decouple at tree level.
- After reduction to 3 dimensions, the vector multiplet scalars take value in a quaternionic-Kähler space $\mathcal{M}_3$, known as the $c-map$ of the special Kähler space $\mathcal{M}_4$.

Ferrara Sabharwal; de Wit Van Proyen Vanderseypen

- The black hole potential splits into two pieces,

$$V_{BH}(p, q; z^i, \bar{z}^i) = |Z|^2 + \partial_i |Z| \, g^{i\bar{j}} \, \partial_{\bar{j}} |Z|$$

where Z is the central charge $Z = e^{K/2}(q_I X^I - p^I F_I)$.

Conserved charges and black hole potential I

- Supersymmetric solutions are obtained by cancelling each term in the kinetic energy against the corresponding term in the potential, leading to the attractor flow equations:

$$\frac{dU}{d\rho} = -e^U |Z| \,, \quad \frac{dz^i}{d\rho} = -2 e^U g_{i\bar{j}} \partial_{\bar{j}} |Z|$$

- The 4D moduli are attracted towards the horizon to the value $z^*_{p,q}$ minimizing $|Z|$ at fixed values of the charges:

$$\mathrm{Re} X^I = p^I \,, \quad \mathrm{Re} F_I = q_I$$

  The attractor point is a local maximum of the potential: BPS trajectories are extremely fine-tuned !

- If $|Z_*| \neq 0$, this is an $AdS_2 \times S_2$ throat, with $S_{BH} = \pi |Z_*|^2$.

Attractor flow and SUSY geodesic motion I

- The above Bogomolny-type argument does not fix the phase in the second attractor equation, and does not guarantee that the solution is supersymmetric.
- A more rigorous procedure is to reduce the full $D = 4$ SUGRA including fermions, and look at BPS solutions of the resulting SUSY mechanics. Shortcut: consider domain walls in N=2 SUGRA + hypers.
- Using the restricted holonomy $Sp(2) \times Sp(2n_V + 2)$, one may show that SUSY trajectories occur when the quaternionic vielbein $V^{A\alpha}$ $(\alpha = 1, 2,\ A = 1, \ldots, 2n_V + 2)$ obtains a null eigenvector:

$$\exists \epsilon_\alpha \ / \ V^{A\alpha}_\mu \, \dot{\phi}^\mu \, \epsilon_\alpha = 0 \quad \Leftrightarrow \quad V^{A[\alpha} V^{\beta]B} = 0$$

Improved SUSY mechanics - HKC and twistors I

- This SUSY mechanics is rather unusual, insofar as the SUSY comes from a triplet of non-integrable complex structures.
- It is possible to remedy this problem by adding 4 real scalar degrees of freedom, extending the QK manifold to its Hyperkähler cone (HKC), or Swann bundle,

$$\mathbb{R}^+ \times S^3 \to HKC \to QK$$

  The spin connection on S^3 is such that the three almost complex structures become integrable. Geodesics on QK lift to $SU(2)$ invariant geodesics on HKC.
- This construction is very natural in the conformal approach to $N = 2$ supergravity.

De Wit Rocek Vandoren

The twistor space

- The relevant information is captured by the twistor space Z, a two-sphere bundle over QK with a Kähler-Einstein metric. The sphere coordinate z keeps track of the Killing spinor, $z = \epsilon_1/\epsilon_2$.
- In the presence of triholomorphic isometries, the geometry of HKC is controlled by a generalized prepotential $G(\eta^L)$,

$$\langle K(v^L, \bar{v}^L, w_L + \bar{w}_L) + x^L(w_L + \bar{w}_L)\rangle_{w+\bar{w}} = \oint \frac{d\zeta}{2\pi i\zeta} G[\eta^L(\zeta)]$$

where η^L is the "projective multiplet"

$$\eta^L = v^L/\zeta + x^L - \bar{v}^L\zeta$$

Hitchin Lindstrom Rocek; De Wit Rocek Vandoren

- When HKC is the Swann bundle of the c-map of a SK manifold, the generalized prepotential is simply obtained from the prepotential F,

$$G(\eta^L, \zeta) = F(\eta^I)/\eta^\flat$$

Rocek Vafa Vandoren

- The inhomogeneous coordinates $\xi^I = v^I/v^\flat, \tilde{\xi}_I = w_I/v^\flat$, $\alpha = w_\flat/v^\flat$ are complex coordinates on Z, adapted to the Heisenberg symmetries:

$$\begin{aligned} \xi^I &= \zeta^I + i\, e^{U+\mathcal{K}(X)/2} \left(z\, \bar{X}^I + z^{-1} X^I \right) \\ \tilde{\xi}_I &= \tilde{\zeta}_I + i\, e^{U+\mathcal{K}(X)/2} \left(z\, \bar{F}_I + z^{-1}\, F_I \right) \\ \alpha &= a + \zeta^I \tilde{\xi}_I - \tilde{\zeta}_I \xi^I \end{aligned}$$

- The coordinates on the base $\mathcal{M}_3$ are $SU(2)$ invariant combinations of $\xi^I, \tilde{\xi}_I, \alpha$.

Neitkze, Pioline, Rocek, Vandoren, to appear

BPS black holes and holomorphic curves

- Upon lifting the geodesic motion to Z, SUSY is preserved iff the momentum is holomorphic in the canonical complex structure on Z, at any point along the trajectory: 1st class constraints !
- Put differently, the SUSY phase space is the twistor space Z, equipped with its Kähler symplectic form. Its dimension is $4n_V + 6$, almost half that of the generic phase space $T^*(\mathcal{M}_3^*)$.
- BPS solutions correspond to holomorphic curves $\xi^I(\rho), \tilde{\xi}_I(\rho), \alpha(\rho)$ at constant $\bar{\xi}^I, \bar{\tilde{\xi}}_I, \bar{\alpha}$, and are algebraically determined by the conserved charges: integrable system !

The Penrose Transform

- At fixed values of $U, z^i, \zeta^I, \tilde{\zeta}_I, a$, the complex coordinates $\xi^I, \tilde{\xi}_I, \alpha$ on Z are holomorphic functions of the twistor coordinate z: the fiber over each point is a rational curve in Z.
- Starting from a holomorphic function Φ on Z, we can produce a function Ψ on QK

$$\Psi(U, z^i, \bar{z}^{\bar{i}}, \zeta^I, \tilde{\zeta}_I, a) = e^{2U} \oint \frac{dz}{2\pi i z} \, \Phi \left[\xi^I(z), \tilde{\xi}^I(z), \alpha(z) \right]$$

satisfying some generalized harmonicity condition:

$$\left(\epsilon^{\alpha\beta} \nabla_{A\alpha} \nabla_{B\beta} - R_{AB} \right) \Psi = 0$$

- This is a quaternionic generalization of the usual Penrose transform between holomorphic functions on CP^3 and conformally harmonic functions on S^4.

Salamon; Baston Eastwood

Outline

The BPS Hilbert space I

- In terms of geodesic motion on the QK base, the classical BPS conditions $V^{A[\alpha}V^{\beta]B} = 0$ become a set of 2nd order differential operators which have to annihilate the wave function Ψ:

$$\left(\epsilon_{\alpha\beta}\nabla^{A\alpha}\nabla^{B\beta} - R^{AB}\right)\Psi = 0$$

- In terms of the twistor space, the BPS condition $p_{\bar{L}} = 0$ requires that Ψ should be a holomorphic function on Z. More precisely, taking the fermions into account, it should be a section of $H^1(Z, \mathcal{O}(-2))$.
- The equivalence between the two approaches is a consequence of the Penrose transform discussed previously.

The BPS Black Hole Wave-Function I

- Ignore fermionic subtleties, and go back to the simple-minded twistor transform

$$\Psi(U, z^i, \bar{z}^I, \zeta^I, \tilde{\zeta}_I, a) = e^{2U} \oint \frac{dz}{2\pi i z}\, \Phi\left[\xi^I(z), \tilde{\xi}^I(z), \alpha(z)\right]$$

- Consider a black hole with $k = 0$: p^I and q_I can be diagonalized simultaneously, and completely determine (up to normalization) the wave function as a coherent state on Z:

$$\begin{aligned}
\Phi &= \exp\left[i(p^I \tilde{\xi}_I - q_I \xi^I)\right] \\
&= \exp\left[i(p^I \tilde{\zeta}_I - q_I \zeta^I) + i e^{U + K(X)/2}(z \bar{W}_{p,q}(\bar{X}) + z^{-1} W_{p,q}(X))\right]
\end{aligned}$$

The BPS Black Hole Wave-Function II

- The integral over z is of Bessel type, leading to

$$\Psi = e^{2U}\, K_0\left(2i\, e^{U}\, |Z_{p,q}|\right)\; e^{i(p^I\tilde{\zeta}_I - q_I\zeta^I)}$$

  This is peaked around the classical attractor points, with slowly damped, increasingly faster oscillations away from them.

- We could have reached this result 36 mins ago, by naively quantizing the attractor flow:

$$\left\{\begin{array}{ccc} p_U & = & -e^U|Z| \\ p_{\bar{z}^{\bar{i}}} & = & -2e^U\partial_{\bar{i}}|Z| \end{array}\right\} \quad \Rightarrow \Psi \sim \exp\left[2ie^U|Z|\right]$$

- Contrary perhaps to expectations, the wave flattens out towards the horizon ! This is because of the large fine-tuning needed to produce a BPS solution.

Relation to the topological amplitude ?

- Before integrating along the fiber, we found that $\Psi_{p,q} \sim \exp[ie^{U+K/2}(z\bar{W} + z^{-1}W)]$, in "rough" agreement with OVV's answer $\Psi_{p,q} \sim \exp(W)$.
- It is unlikely that Ψ_{top} can be identified as a black hole wave function: it naturally depends on $n_V + 1$ variables, while Ψ_{BH} depends on $2n_V + 3$ variables.
- Instead, the "super-BPS" Hilbert space of tri-holomorphic functions on HKC is the natural habitat of a one-parameter generalization of the topological string amplitude...

Gunaydin Neiztke BP

Outlook

- Higher derivative corrections remain to be incorporated: higher derivative scalar interactions on QK space.
- Multi-centered configurations can be described by certain harmonic maps from $\mathbb{R}^3$ to QK: does that correspond to “second quantization”, i.e. including vertices ?
- For $N \geq 4$, this suggests that the 3D U-duality group controls the BH spectrum: can one obtain the exact degeneracies as Fourier coefs of some “BPS automorphic forms” ? Improve on DVV.
- The equivalence between BH attractor flow and geodesic flow on QK is a reflection of mirror symmetry. Can this be used to compute instanton corrections on hypermultiplet moduli space ?