Curved space-times and degrees of freedom in matrix models

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There are several types of matrix models, but here for the sake of concreteness we consider IIB matrix model.

Abstract

IIB matrix model is a candidate for the constructive definition of string theory. It is nothing but the large-N reduced model of 10D super Yang-Mills theory,

$$S = -\frac{1}{g^2} Tr\left(\frac{1}{4}[A^\mu, A^\nu]^2 + \frac{1}{2}\overline{\Psi}\Gamma^\mu [A^\mu, \Psi]\right)$$

A^μ ($\mu = 1 \sim 10$),

Ψ (10D Majorana-Weyl) : N×N hermitian

This theory seems to describe fluctuations around a flat space-time. The basic question is whether it can describe curved space-times, and it really contains invariance of the general relativity, diffeomophism and local Lorentz invariance, or not. Here, we will show that indeed it is the case, if we introduce a *new interpretation* for the matrices.

Matrix model contains space time.

First of all let us see that the matrix model contains space-time in its degrees of freedom.

IIB matrix model is formally obtained from 10D SU(N) super Yang Mills theory by a dimensional reduction to zero dimensions. Therefore, if we think naively, this reduction severely truncates the degrees of freedom. However it is not the case if N is infinitely large. In fact in this case, the matrix model *contains* the original 10D model.

10D super YM $\rightarrow$ IIB matrix model $\supset$ 10D super YM

dimensional reduction to 0D $\quad N \to \infty$

The reason is as follows:

The dynamical variables A_μ's are $N \times N$ matrices, but let us treat them in a little abstract manner. That is, we regard $N \times N$ matrices as linear transformations on some linear space V:

$$A_\mu \in End(V), \quad V \cong \mathbb{C}^N .$$

So, if N is infinitely large, V is an infinite dimensional vector space, and we can take various expressions for V.

First let us assume V is the space of n-component complex scalar fields on 10D space-time:

$$V = \{\varphi^i(x) : \mathbb{R}^{10} \to \mathbb{C}^n\} .$$

Then a matrix, a linear transformation on V, is nothing but an integration kernel, or a bilocal field $\kappa^{i\,j}(x,y)$:

$$T \in End(V),$$

$$(T\varphi)^i(x) = \int d^{10}y \sum_{j=1}^{n} \kappa^{i\,j}(x,y)\varphi^j(y)$$

$$= c_{(0)}{}^{i\,j}(x)\,\varphi^j(x) + c_{(1)}^{\mu}{}^{\,i\,j}(x)\,\partial_\mu\varphi^j(x) + c_{(2)}^{\mu\nu}{}^{\,i\,j}(x)\,\partial_\mu\partial_\nu\varphi^j(x) + \cdots$$

At least formally, such bilocal field can be expanded as a differential operator of infinite order whose coefficients are $n \times n$ matrices.

In particular as a special value of A_μ, we can take the covariant derivative

$$A_\mu = i\, D_\mu = i\left(\delta^{ij}\partial_\mu + i a_\mu{}^{ij}(x)\right) \in End(V),$$

which means the whole space of gauge field configurations is embedded in the space of matrices $End(V)$.

Thus we have seen that the dynamical degrees of freedom of the original 10D theory are completely included in the matrix model if we take the large-N limit:

$$\text{IIB matrix model} \quad \supset \quad \text{SU(n) 10D super YM.}$$
$$(N \to \infty)$$

Furthermore in this embedding the local gauge symmetry is realized as a part of the SU(N) symmetry of the matrix model:

$$\delta A_\mu = i\left[\lambda, A_\mu\right], \lambda \in End(V) : \text{0-th order differential operator} .$$

Large-N reduction

In this sense, IIB matrix model has more dynamical degrees of freedom than 10D super Yang-Mills theory. However, it turns out that the difference between them is *not* so large. This fact is known as "the large-N reduction".

For example, it is known that

(1) If we quench the diagonal elements of A_μ,

$$A_\mu = P_\mu + \tilde{A}_\mu,$$

$$P_\mu\text{: diagonal and fixed, } \tilde{A}_\mu\text{: offdiagonal},$$

the matrix model is equivalent to the large-N Yang-Mills theory.

(2) If we expand A_μ around a noncommutative background

$$A_\mu = \hat{p}_\mu + a_\mu(\hat{x}),$$

$$\left[\hat{p}_\mu, \hat{p}_\nu\right] = iB_{\mu\nu},$$

$$\hat{x}^\mu = C^{\mu\nu}\hat{p}_\nu,\ B_{\mu\nu}C^{\nu\lambda} = \delta_\mu^\lambda,$$

and constrain the fluctuation that

$$\left|a_\mu(x)\right| < \infty\ (\left|x\right| \to \infty),$$

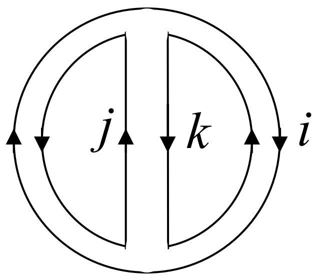
the matrix model is equivalent to Yang-Mills theory in a noncommutative space.

More presicely,

<u>(1) quenched reduced model</u>

Parisi (1982)
Gross, Kitazawa
Bhanot, Heller, Neuberger
Das, Wadia

The path integral for the off-diagonal elements of A_μ can be expressed by such Feynman diagrams as



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And if the eigenvalues p_i^{μ} of P^{μ} are uniformly distributed in the momentum space, the summation over the indices becomes the integration over loop momenta in the large-N limit:

$$\sum_{i,j,k} \sum_{(i\neq j, j\neq k, k\neq i)} \sum \frac{1}{\left(p_i - p_j\right)^2 \left(p_j - p_k\right)^2 \left(p_i - p_k\right)^2}$$

$$\rightarrow \int\int\int d^{10}k_1 \, d^{10}k_2 \, \frac{1}{k_1{}^2 k_2{}^2 (k_1 + k_2)^2}$$

Therefore the quenched reduced model becomes equivalent to the original 10D field theory in the large-N limit.

It is useful to compare this result with the expression in terms of the differential operators. If we express A_μ as a differential operator

$$A_\mu = c_\mu^{(0)i\,j}(x) + c_\mu^{(1)\,\nu\;i\,j}(x)\,\partial_\nu + c_\mu^{(2)\,\nu\,\lambda\;i\,j}(x)\partial_\nu\partial_\lambda + \cdots,$$

the background of the quenched reduced model is nothing but

$$A_\mu^{(0)} = i\partial_\mu\,.$$

So, in the classical level, fluctuations from this background consist of infinitely many fields of various higher spins. Because diagonal elements are nothing but polynomials of $i\partial_\mu$, the condition of quenching suppresses the zero mode of each field. However we still have very large degrees of freedom as far as we consider classical dynamics.

On the other hand the quenched reduced model tells us that these degrees of freedom are combined to *one vector field*, if we treat them quantum mechanically.

(2) twisted reduced model

Gonzalez-Arroyo, Okawa (1983)
Gonzalez-Arroyo, Korthals Altes
Rediscovered in the context of NCFT.

We pick up a classical solution of the equation of motion

$$\left[A_\mu,\left[A_\mu,A_\nu\right]\right]=0$$

that is given by the CCR with $10/2=5$ degrees of freedom,

$$A_\mu^{(0)}=\hat{p}_\mu\otimes 1_n$$

$$\left[\hat{p}_\mu,\hat{p}_\nu\right]=i\,B_{\mu\nu}\ \ (B_{\mu\nu}\in\mathbb{R}),\quad 1_n:\ \mathrm{n}\times\mathrm{n\ unit\ matrix}\cdot$$

and expand the dynamical variables around it:

$$A_\mu=A_\mu^{(0)}+\hat{a}_\mu,\quad \psi=0+\hat{\psi}\ .$$

Next we introduce the following correspondence between operators and functions:

$$\hat{o}=\int\frac{d^{10}k}{(2\pi)^{10}}\tilde{o}(k)\exp(ik_\mu\hat{x}^\mu)\Leftrightarrow\quad o(x)=\int\frac{d^{10}k}{(2\pi)^{10}}\tilde{o}(k)\exp(ik_\mu x^\mu)$$

$$B_{\mu\nu}C^{\nu\lambda}=\delta_\mu^\lambda,\quad \hat{x}^\mu=C^{\mu\nu}\hat{p}_\nu\,.$$

($\hat{o}$ is the Weyl ordering of $o(\hat{x})$.)

Then we can show the following mapping rules:

$$\left[\hat{p}_\mu, \hat{o}\right] \leftrightarrow i\partial_\mu o,$$
$$\hat{o}_1\hat{o}_2 \qquad \leftrightarrow o_1 * o_2,$$
$$Tr\left(\hat{o}\right) = \frac{\sqrt{\det B}}{\left(2\pi\right)^5}\int d^{10}x\, o(x)$$

Using them, we can rewrite the matrix model action as that of a field theory on the noncommutative space-time:

$$S_{IIB} = \frac{\sqrt{\det \mathrm{B}}}{\left(2\pi\right)^5}\int d^{10}x\ \mathrm{tr}\left(-\frac{1}{4}F_{\mu\nu}{}^2 + \frac{1}{2}\bar{\psi}\gamma^\mu\left[D_\mu, \psi\right]\right)_*.$$

Again we have seen that the matrix model is “almost equivalent” to the 10D field theory.

matrix model contains gravity

So far, we have seen that IIB matrix model is “slightly larger” than the 10D super Yang-Mills theory.

On the other hand it is known that IIB matrix model also contains gravity, if we consider a different kind of background.

Ishibashi, Kitazawa, Tsuchiya and HK (1996)

We consider the linear space V as before:

$$V = \{\varphi^i(x) : \mathbb{R}^{10} \to \mathbb{C}^n\},$$

but this time we consider a background $A^{(0)}{}_\mu$ that is diagonal in the *x-space.* In terms of integration kernel, it is expressed as

$$A^{(0)}{}_\mu = x^\mu \delta^{(10)}(x-y)\delta^{ij}.$$

Or in term of differential operator

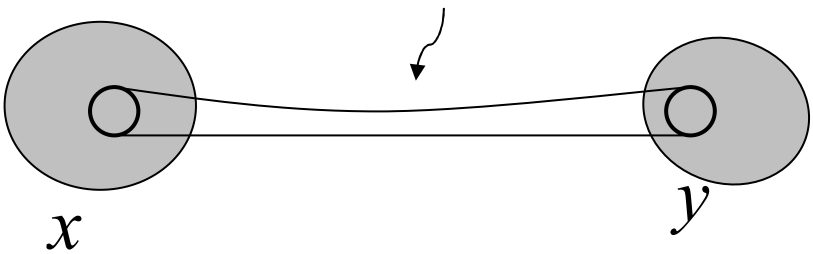
$$A^{(0)}{}_\mu = x^\mu \delta^{ij}.$$

If we expand A_μ around it

$$A_\mu = A^{(0)}{}_\mu + \tilde{A}_\mu,$$

the one-loop effective Lagrangian contains terms that correspond to the exchange of graviton and dilaton:

$$S_{eff} = - \iint d^{10}x\, d^{10}y \frac{1}{(x-y)^8} \{ \, const \cdot \mathrm{tr}(f_{\mu\lambda}(x) f_{\nu\lambda}(x)) \;\; \mathrm{tr}(f_{\mu\rho}(y) f_{\nu\rho}(y))$$
$$- const \cdot \mathrm{tr}(f_{\mu\nu}(x) f_{\mu\nu}(x)) \;\; \mathrm{tr}(f_{\lambda\rho}(y) f_{\lambda\rho}(y)) \;\; + \cdots \}$$



Question:

We have seen that Yang-Mills theory is embedded in IIB matrix model, and in this embedding the local gauge symmetry is a part of the SU(N) symmetry of the matrix model.
Furthermore we have seen that IIB matrix model contains gravity.

Can we find a good way of embedding gravity in IIB matrix model, in such a way that diffeomorphism and local Lorentz invariance become a part of the SU(N) symmetry of the matrix model?

More concretely, suppose we have a D-dimensional curved space M and the covariant derivative $\nabla_a \ (a=1\sim D)$ on it.

Find

(i) a good space V and

(ii) a good object $\nabla_{(a)}$ which is equivalent to ∇_a

such that each component of $\nabla_{(a)}, (a=1..10)$ is expressed as a linear transformation on V.

Difficulty

The covariant derivative $\nabla_a \ (a=1\sim D)$ maps a scalar field to a vector field, and a vector field to a tensor field, and so on. Therefore, it can *not* simply be regarded as a linear transformation on some space such as

$$A_a = i\nabla_a \, .$$

The difficulty may become clearer, if we compare the product $\nabla_a\nabla_b$ and A_aA_b . In $\nabla_a\nabla_b$, the spin connection contained in ∇_a mixes the index b . On the other hand A_aA_b simply means the product of A_a and A_b . Therefore the equation like $A_a = i\nabla_a$ can not hold.

Answer:

Suppose $M = \bigcup_I U_I$ is a D-dim Riemannian manifold with a spin structure, and we have a transition function on each overlapping region

$$t_{IJ}\left(x\right) \in spin(D),\, x \in U_I \cap U_J \quad .$$

Here we assume all indices are Lorentz indices, and in particular, the covariant derivative is expressed as

$$\nabla_a = e_a^{\mu}\left(\partial_{\mu} + \omega_{\mu}^{bc} O_{bc}\right),$$

where O_{bc} is the Lorentz generator.

(i) $V = C^{\infty}(E_{prin})$

First we consider the principal $spin(D)$ bundle E_{prin} on M associated with the spin structure. In other words, we consider a direct product $U_I \times Spin(D)$ for each patch U_I, and glue them together by the following rule:

For $x \in U_I \cap U_J$,

$$(x, g_I) \in U_I \times Spin(D) \sim (x, g_J) \in U_J \times Spin(D)$$
$$\Leftrightarrow g_I = t_{IJ}(x) g_J$$

Then we take the space of functions on E_{prin} as V. In other words, we regard matrices as bilocal fields or differential operators on E_{prin}.

(ii) $\nabla_{(a)} = R_{(a)}{}^{b}\left(g^{-1}\right)e_b^{\mu}\left(\partial_\mu + \omega_\mu^{cd}(x)\hat{O}_{cd}\right) \in End(C^\infty(E_{prin}))$

Let $R_a{}^b\left(g\right)$ be the rep. matrix of the vector rep. of $Spin(D)$, and $\hat{O}_{ab}$ be the infinitesimal left action on the function space on $Spin(D)$: ($\hat{O}_{ab}$ is the derivative along the fiber.)

$$\varepsilon^{ab}\hat{O}_{ab}\varphi(g) = \varphi((1-\varepsilon^{ab}\tau_{ab})g) - \varphi(g).$$

Then we define a differential operator on E_{prin} by

$$\nabla_{(a)} = R_{(a)}{}^{b}\left(g^{-1}\right)e_b^{\mu}\left(\partial_\mu + \omega_\mu^{cd}(x)\hat{O}_{cd}\right).$$

We can show that each component of $\nabla_{(a)}, (a=1..10)$ is a globally defined differential operator on E_{prin}, and thus a linear transformation on V. Therefore it can be expressed by a matrix. (Proof)

$$\begin{aligned}
\nabla_{(a)}^{[I]} &= R_{(a)}{}^{b}\left(g_I{}^{-1}\right)\nabla_b^{[I]} \\
&= R_{(a)}{}^{b}\left(g_I{}^{-1}\right)R_b{}^{c}\left(t_{IJ}(x)\right)\nabla_c^{[J]} \\
&= R_{(a)}{}^{c}\left(g_I{}^{-1}t_{IJ}(x)\right)\nabla_c^{[J]} \\
&= R_{(a)}{}^{c}\left(\left(t_{IJ}(x)^{-1}g_I\right)^{-1}\right)\nabla_c^{[J]} \\
&= R_{(a)}{}^{c}\left(g_J{}^{-1}\right)\nabla_c^{[J]} \\
&= \nabla_{(a)}^{[J]}
\end{aligned}$$

Furthermore we can show that each component of $\nabla_{(a)}$ is hermitian for the natural measure on E_{prin}:

$$(f,h) = \int_{E_{prin}} f^* h = \int d^D x\sqrt{g}\int dg\; f(x,g)^* h(x,g) \,.$$

Example S_2 with homogeneous metric

$$spin(2)=\{e^{i\theta};0\le\theta<2\pi\}\simeq S_1$$

$$E_{prin}\simeq S_1 \ bundle\ over\ S_2\simeq S_3$$ (Hopf bundle)

$$V=C^{\infty}(E_{prin})=\{\varphi(z,\theta);S_3\to\mathbb{C}\}$$

z: the stereographic coordinate of S_2

$$\nabla_{(+)}=e^{-2i\theta}\{(1+z\bar{z})\partial_z+\frac{i}{2}\bar{z}\partial_\theta\},\qquad \leftarrow \qquad \nabla_+=(1+z\bar{z})\partial_z+\frac{i}{2}\bar{z}O_{12}$$

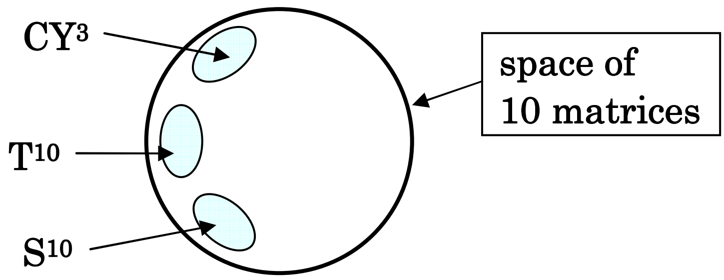
$$\nabla_{(-)}=e^{2i\theta}\{(1+z\bar{z})\partial_{\bar{z}}-\frac{i}{2}z\partial_\theta\},\qquad \nabla_-=(1+z\bar{z})\partial_{\bar{z}}-\frac{i}{2}zO_{12}$$

where $\pm=1\pm i2$.

Each of $\nabla_{(+)}$ and $\nabla_{(-)}$ is a globally defined differential operator on E_{prin}.

In this manner, S_2 is realized in terms of two matrices. This should be distinguished from the ordinary fuzzy sphere, which is obtained by embedding to the space of three matrices.

Similarly, any Riemannian manifold with dimension less than or equal to D can be coded in the space of D matrices.



What is the space V ?

$$V = \bigoplus_{r:\text{ rep. of Spin(D)}} \underbrace{(V_r \oplus \cdots \oplus V_r)}_{d_r \text{ (dim. of } r \text{)}}, \quad V_r : \text{space of a field with rep. } r.$$

scalar, spinor, vector,…etc.

($\because$) Since an element of V is a function on E_{prin}, at each point on M, it gives a function $Spin(D) \to \mathbb{C}$.

In general, the space of functions on a group G forms a special representation called the regular representation, which is isomorphic to

$$v_{reg} \cong \bigoplus_{r:\text{rep. of G}} \underbrace{(v_r \oplus \cdots \oplus v_r)}_{d_r},$$

where v_r is a representation of G, and d_r is its dimension.

A function $G \to \mathbb{C}$ can be expanded as

$$f(g) = \sum_{r,i,j} c_{<r>}{}^{i,j}\, R_{<r>}{}^{i,j}(g),$$

where $R_{<r>}{}^{i,j}(g)$ is the rep. matrix for r.

The action of an element h of G on f is assumed to be

$$\begin{aligned} f(h^{-1}g) &= \sum_{r,i,j} c_{<r>}{}^{i,j}\, R_{<r>}{}^{i,j}(h^{-1}g) \\ &= \sum_{r,i,j} c_{<r>}{}^{i,j}\, R_{<r>}{}^{i,k}(h^{-1}) R_{<r>}{}^{k,j}(g) \\ &= \sum_{r,i,j} c_{<r>}{}^{k,j}\, R_{<r>}{}^{k,i}(h^{-1}) R_{<r>}{}^{i,j}(g). \end{aligned}$$

Therefore,

$$f(g) \mapsto f'(g) = f(h^{-1}g) \Leftrightarrow c_{<r>}{}^{i,j} \mapsto c'_{<r>}{}^{i,j} = R_{<r>}{}^{k,i}(h^{-1}) c_{<r>}{}^{k,j}.$$

The regular rep. has the following remarkable property:

$$v_{reg} \otimes v_r \cong v_{reg} \oplus \cdots \oplus v_{reg} \text{ , for any } r.$$

More explicitly, this isomorphism is given by

$$f^i(g) \in v_{reg} \otimes v_r \mapsto f^{(i)}(g) = R_{<r>}{}^{ij}(g^{-1}) f^j(g).$$

Then $f^{(i)}(g)$ transforms under $h \in G$ as

$$f^{(i)}(g) \mapsto$$
$$R_{<r>}{}^{ij}(g^{-1}) R_{<r>}{}^{jk}(h) f^k(h^{-1}g) = R_{<r>}{}^{ik}(g^{-1}h) f^k(h^{-1}g) = f^{(i)}(h^{-1}g).$$

Therefore we have

$$\nabla_a : V \to V \otimes T \cong V \oplus \cdots \oplus V,$$

where T is the tangent bundle, and the combined map is given by

$$\nabla_{(a)} = R_{(a)}{}^b \left(g^{-1} \right) e_b^\mu \left(\partial_\mu + \omega_\mu^{cd}(x) \hat{O}_{cd} \right).$$

New interpretation of IIB matrix model

We now regard the matrices in IIB matrix model as linear transformations on $C^\infty(E_{prin})$.

Here we consider the classical EOM derived from the action

$$S = -\frac{1}{4} Tr\left(\left[A_a, A_b\right]^2\right) + fermions \, .$$

If we set the fermions to be zero, it becomes

$$\left[A_a\left[A_a, A_b\right]\right] = 0 \, .$$

Now we can impose the following Ansatz

$$A_a = i\nabla_{(a)} \, ,$$

because $\nabla_{(a)}$ is a well defined linear transformation, and we have

$$\left[\nabla_{(a)}\left[\nabla_{(a)}, \nabla_{(b)}\right]\right] = 0 \, .$$

Let's rewrite this equation in terms of the ordinary covariant derivative ∇_a.

Formula

$$\nabla_{(a)}\nabla_{(b)} = R_{(a)}{}^{c}\left(g^{-1}\right)\nabla_c \, R_{(b)}{}^{d}\left(g^{-1}\right)\nabla_d$$
$$= R_{(a)}{}^{c}\left(g^{-1}\right) R_{(b)}{}^{d}\left(g^{-1}\right)\nabla_c\nabla_d$$

In the last expression, the Lorentz generator in ∇_c acts on the index d of ∇_d.

Using this, we have

$$0 = \left[\nabla_{(a)}, \left[\nabla_{(a)}, \nabla_{(b)}\right]\right]$$
$$\Leftrightarrow$$
$$0 = \left[\nabla_a, \left[\nabla_a, \nabla_b\right]\right]$$
$$= \left[\nabla_a, R_{ab}{}^{cd} O_{cd}\right] = (\nabla_a R_{ab}{}^{cd}) O_{cd} - R_{ab}{}^{ca}\nabla_c$$
$$\Leftrightarrow \nabla_a R_{ab}{}^{cd} = 0 \, , \, R_{ab} = 0$$
$$\Leftrightarrow R_{ab} = 0 \, .$$

The Einstein equation follows from the EOM of IIB matrix model.

If we start with IIB action with a mass term

$$S' = -\frac{1}{4} Tr\left(\left[A_a, A_b\right]^2\right) + \frac{m^2}{2} Tr\left(A_a{}^2\right) + fermions ,$$

we have the Einstein equation with a cosmological constant

$$R_{ab} = -m^2 \delta_{ab} \quad .$$

Diffeomorphism and local Lorentz invariance

We now show that the symmetries of the general relativity are realized as parts of the SU(N) symmetry of the matrix model.

In the matrix model the infinitesimal SU(N) symmetry is given by

$$\delta A_a = [\Lambda, A_a], \Lambda \in End(V).$$

If we interpret V as $C^\infty(E_{prin})$, we can take various Λ from $End(C^\infty(E_{prin}))$.

(1) diffeomorphism

$$\Lambda = \frac{1}{2}\{\lambda^{(a)}(x,g), \nabla_{(a)}\} \in End(C^\infty(E_{prin})) \sim \lambda^a(x)\nabla_a$$

$$\lambda_{(a)}(x,g) = R_{(a)}{}^b\left(g^{-1}\right)\lambda_b(x)$$

$\Rightarrow \delta A_a = [\Lambda, A_a]$ correctly reproduces the diffeomorphism.

(2) local Lorentz

$$\Lambda = \lambda^{ab}(x)\hat{O}_{ab} \in End(C^\infty(E_{prin}))$$

$\Rightarrow \delta A_a = [\Lambda, A_a]$ is the local Lorentz transformation.

Summary

- Any D-dimensional manifold can be embedded in D matrices.
- Accordingly, the matrices in IIB matrix model can be interpreted as differential operators on the principal bundle on any manifold of less than or equal to 10 dimensions.
- The classical EOM of IIB matrix model gives the Einstein equation.
- So far we have analyzed classical EOM. However, as we have seen for the flat space, it is expected that the degrees of freedom become much smaller in the quantized level. It is important to consider what remains in the quantized level.

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- In order to implement SUSY, we can consider super manifold instead of the ordinary manifold, and consider the matrices as differential operators on the associated principal bundle. Then V becomes a super vector space and the matrices should be regarded as super matrices, but the form of the action of IIB matrix model seems to work without any modification.