Phase Transitions with α' Correction

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ISM06-Puri

Outline

Euclidean spaces

$$\begin{array}{cc} \text{Bulk} & \text{Boundary} \\ AdS_5 & S^4 \\ \downarrow & \downarrow \\ \text{X} & \text{M} \end{array}$$

$$Z_{CFT}(M) = \sum_i exp(-N^2 F(X_i))$$
$$I_s = N^2 F(X)$$

Euclidean spaces

Bulk
AdS_5

Boundary
S^4

Thermal AdS

X $\qquad$ $M \rightarrow S^1 \times S^3$

AdS Schwarzschild

$$Z_{CFT}(M) = \sum_i exp(-N^2 F(X_i))$$
$$I_s = N^2 F(X)$$

Thermal AdS

- The metric

$$ds^2 = \left(\frac{r^2}{l^2} + 1\right) dt^2 + \left(\frac{r^2}{l^2} + 1\right)^{-1} dr^2 + r^2 d\Omega^2$$

- t is periodic with any periodicity.
- l is related to the cosmological constant.

AdS Schwarzschild Black Hole

- The metric

$$ds^2 = -(1 + \frac{r^2}{l^2} - \frac{m}{r^2})\, dt^2 + (1 + \frac{r^2}{l^2} - \frac{m}{r^2})^{-1} dr^2 + r^2 d\Omega^2$$

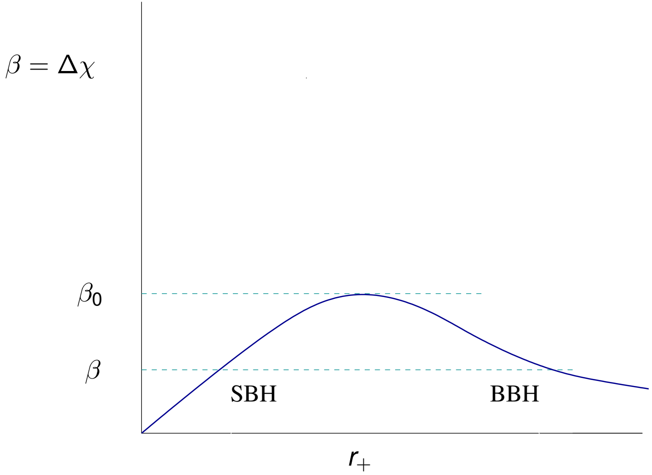
- Asymptotically AdS
- Horizon at $r_\pm$

$$1 + \frac{r^2}{l^2} - \frac{m}{r^2} = 0$$

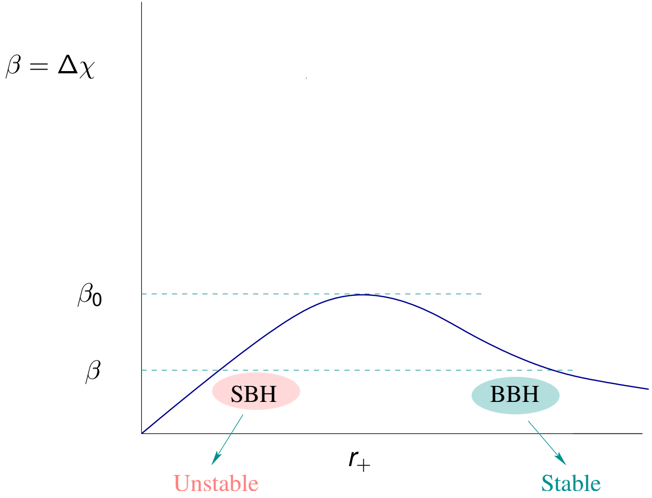
- To avoid conical singularity

$$\Delta\chi = \frac{2\pi r_+ l^2}{2r_+^2 + l^2}$$

AdS Schwarzschild Black Hole



AdS Schwarzschild Black Hole



Free Energy

- Free energy from action

$$\beta F = I_X$$

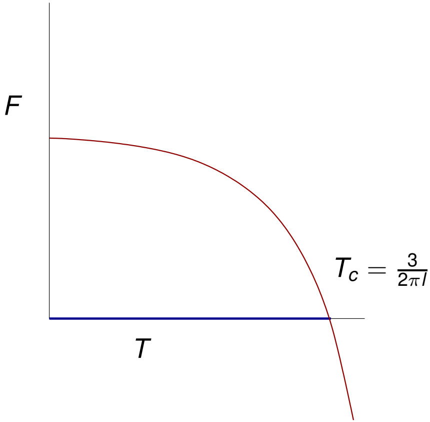
The free energy in general diverges for the individual metrics.

- Compute the difference $I(X_1) - I(X_2)$
- Free energy of Thermal AdS is zero.
- Free energy of the AdS Sch. Black Holes.

$$F = \frac{2\pi^2 r_+^2}{\kappa_5}\left(1 - \frac{r_+^2}{l^2}\right)$$

- There is a transition when $r_+ = l$

Hawking-Page Transition



Gauss-Bonnet black holes

$$I = \int d^{n+1}x\sqrt{-g_{n+1}}\left[\frac{R}{\kappa_{n+1}} - 2\Lambda + \alpha(R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd})\right]$$

This action possesses black hole solutions.

- Solution for Metric

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{V(r)} + r^2 d\Omega_{n-1}^2,$$

$$V(r) = 1 + \frac{r^2}{2\hat{\alpha}} - \frac{r^2}{2\hat{\alpha}}\left[1 - \frac{4\hat{\alpha}}{l^2} + \frac{4\hat{\alpha}m}{r^n}\right]^{\frac{1}{2}}$$

- $d\Omega_{n-1}^2$ is the metric of a $n-1$ dimensional sphere.

$$V(r) = 1 + \frac{r^2}{2\hat{\alpha}} - \frac{r^2}{2\hat{\alpha}}\left[1 - \frac{4\hat{\alpha}}{l^2} + \frac{4\hat{\alpha}m}{r^n}\right]^{\frac{1}{2}}$$

- $\hat{\alpha} = (n-2)(n-3)\alpha\kappa_{n+1}$
- $l^2 = -n(n-1)/(2\kappa_{n+1}\Lambda)$
- Asymptotic behavior

$$V(r) = 1 + \left[\frac{1}{2\hat{\alpha}} - \frac{1}{2\hat{\alpha}}\left(1 - \frac{4\hat{\alpha}}{l^2}\right)^{\frac{1}{2}}\right]r^2$$

- Horizon fo $n = 4$ (five dimensions)

$$r^2 = r_+^2 = \frac{l^2}{2}\left[-1 + \sqrt{1 + \frac{4(m - \hat{\alpha})}{l^2}}\right]$$

Thermodynamics

Thermodynamics of these black holes can be obtained by standard Euclidean action calculation.

- Free Energy and Temperature

$$F = \frac{\omega_{n-1} r_+^{n-4}}{\kappa_{n+1}(n-3)(r_+^2+2\hat{\alpha})}\Big[(n-3)r_+^4(1-\frac{r_+^2}{l^2}) - \frac{6(n-1)\hat{\alpha} r_+^4}{l^2} + (n-7)\hat{\alpha} r_+^2 + 2(n-1)\hat{\alpha}^2\Big],$$

$$T = \frac{(n-2)}{4\pi r_+(r_+^2+2\hat{\alpha})}\Big[r_+^2 + \frac{n-4}{n-2}\hat{\alpha} + \frac{n}{n-2}\frac{r_+^4}{l^2}\Big]$$

- Entropy

$$S = \frac{4\pi\omega_{n-1} r_+^{n-1}}{\kappa_{n+1}}\Big[1 + \frac{n-1}{n-3}\frac{2\hat{\alpha}}{r_+^2}\Big]$$

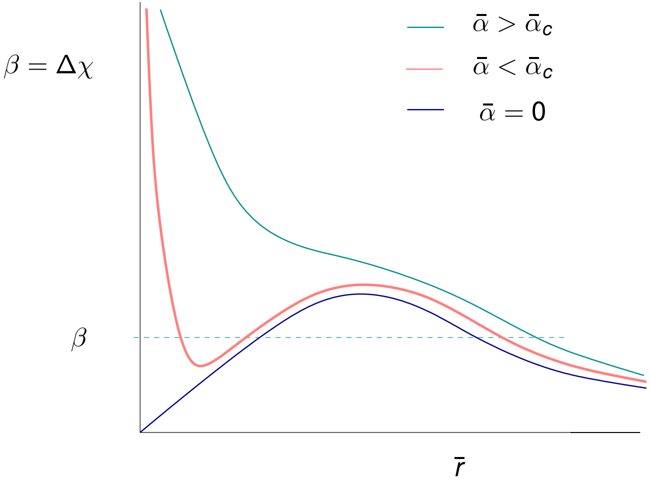


Figure: β as a function of $\bar{r}$ for different values of $\bar{\alpha}$.

Free energy vs radius

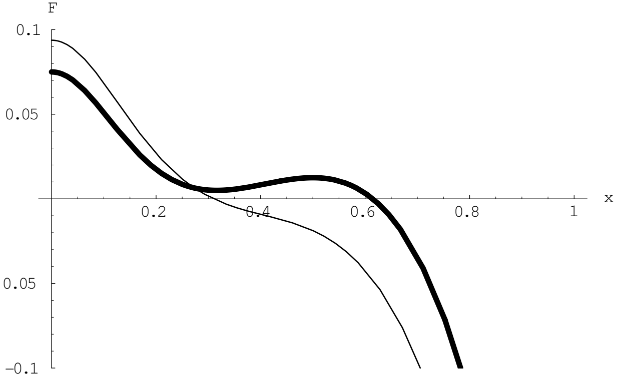


Figure: Free energy as a function of $x = \bar{r}$ for different values of $\bar{\alpha}$. The thicker line is for $\bar{\alpha} = 1/40$ and the other one $\bar{\alpha} = 1/32$.

Free energy vs Temperature

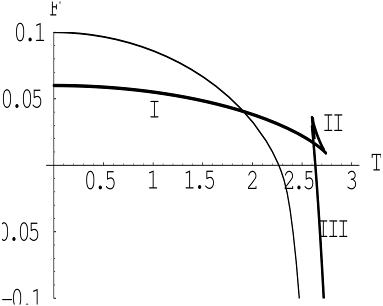


Figure: Free energy as a function of temperature. The thicker one is for $\bar{\alpha} = 1/50$ while the other one is for $\bar{\alpha} = 1/30$.

Phases

Different phase structures as we vary $\bar{\alpha}$.

1. $\bar{\alpha} \leq \bar{\alpha}_c$: Three branches:
   - I+III: Specific heat positive $\rightarrow$ Stable
   - II : Specific heat negative $\rightarrow$ Unstable
   - HP1 : I $\rightarrow$ III
   - HP2 : Thermal AdS $\rightarrow$ III

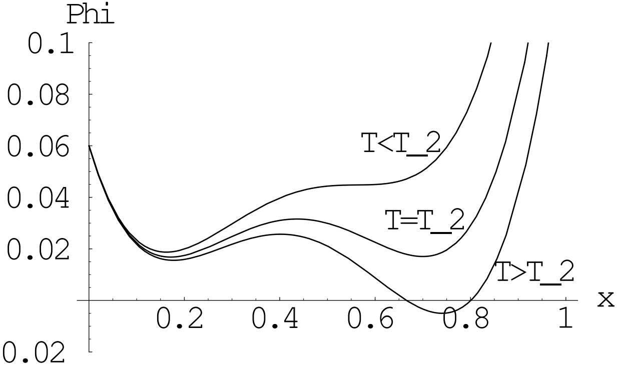
$$T_c = \frac{3}{2\pi l} - \frac{33\bar{\alpha}}{4\pi l} + \mathcal{O}(\bar{\alpha}^2)$$

2. $\bar{\alpha} > \bar{\alpha}_c$: Only one branch for stable Black hole.

Phases

- Landau function around the critical point: $\bar{\alpha} = 1/50$

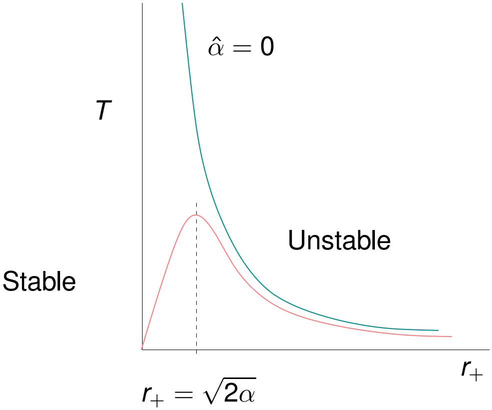
$$\Phi(T, \bar{r}) = \frac{\omega_3 l^2}{\kappa_5}(3\bar{r}^4 - 4\pi l T \bar{r}^3 + 3\bar{r}^2 - 24\pi\bar{\alpha} l T \bar{r} + 3\bar{\alpha})$$



Some comments on the flat case

- In the limit $l \to \infty$ ($\Lambda = 0$) the solution reduces to the asymptotically flat Gauss-Bonnet black hole.
- The temperature is given by

$$T = \frac{r_+}{2\pi(r_+^2 + 2\hat{\alpha})}$$

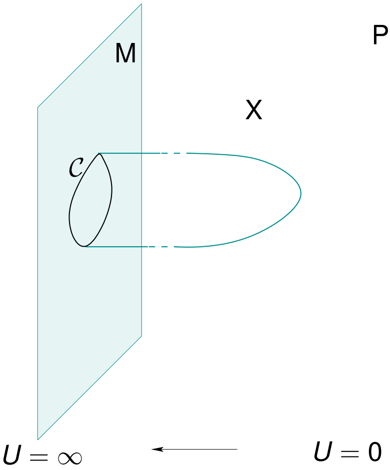

$\hat{\alpha} = 0$
T
Unstable
Stable
r_+
$r_+ = \sqrt{2\alpha}$

Gauge theory

- $SU(N)$, N=4 SYM Theory on $S^1 \times S^3$ in the limit $N \to \infty$.
- Identify Hawking-Page transition in the bulk with Confinement/Deconfinement transition on the boundary.
- Order parameter - Wilson loop operator

$$W(\mathcal{C}) = \frac{1}{N} \mathrm{Tr} P \exp \int_{\mathcal{C}} A$$

Wilson Loop



Proposal (Maldacena,Rey,Yee)

$$\langle W(\mathcal{C}) \rangle = e^{-S}$$

Free Energy

- Another order parameter for deconfined hase is the Free Energy.
- Pure $U(N)$ gauge theory contains $N^2 - 1 \sim N^2$ gluons. For the decofined phase we expect $F \sim \mathcal{O}(N^2)$ and for cofined phase we expect $F \sim \mathcal{O}(1)$.
- Gravity computation (Free energy of black hole) corresponding to the strongy coupled gauge theory ($g_{YM}^2 N = \lambda \to \infty$)

$$F = -\frac{\pi^2}{6} N^2 T^4 \left(\frac{3}{4} + \frac{45}{32} \frac{\zeta(3)}{(2\lambda)^{3/2}} \right)$$

Free Energy

- Gauge theory weak coupling (λ) expansion (free gluons)

$$F = -\frac{\pi^2}{6} N^2 T^4 \left(1 - \frac{3\lambda}{2\pi^2}\right)$$

- Apart from the dependence on the coupling λ the free energy scales in the same way w.r.t N^2 and T for both ends of the coupling.

Effective theory

- Partition function (Aharony et al.)

$$Z(\lambda, T) = e^{-\beta F} = \int \mathcal{D}A e^{-S_{YM}(A)}$$

- Evaluate at weak coupling $\lambda \to 0$. Free gauge theory.
- Gauge condition

$$\begin{aligned} \partial_i A^i &= 0 \\ \partial_t \alpha(t) &= 0 \end{aligned}$$

$$\alpha = \frac{1}{V_3} \int_{S^3} A_0$$

Matrix model

- Effective Action

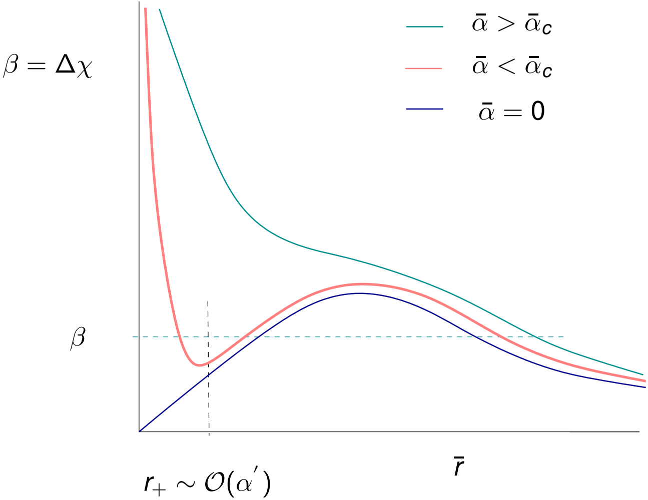
$$Z(\lambda, T) = e^{-\beta F} = \int [dU] e^{-S_{eff}(U)}$$
$$U = e^{i\beta\alpha}$$

$$\frac{1}{N}\langle U\rangle \rightarrow \text{Wilson loop}$$

- Measure

$$[dU] = \prod_{i,j}[dU_{i,j}] = \prod_{i} d\lambda_i \prod_{i<j} \sin^2\left(\frac{\beta(\lambda_i - \lambda_j)}{2}\right)$$

Two possibilities



A simple model

- Quartic potential (Alvarez-Gaume et al.)

$$Z(\lambda, T) = \int [dU] \exp \left[a \mathrm{tr} U \mathrm{tr} U^\dagger + b/N^2 \left(\mathrm{tr} U \mathrm{tr} U^\dagger \right)^2 \right]$$

- a and b are functions of λ and T.
- Study the phases of this model.

Phases

- Saddle-point equation

$$
\begin{aligned}
a\rho + 2b\rho^3 &= \rho & 0 \leq \rho \leq \frac{1}{2} \\
&= \frac{1}{4(1-\rho)} & \frac{1}{2} \leq \rho \leq 1
\end{aligned}
$$

  where $\rho^2 = (1/N^2)\mathrm{tr}U\mathrm{tr}U^\dagger$.
- Existence of solution for ρ.
- $\rho = 0$ is always a solution.

Comparison with Gravity

- Phases equivalent to strongly coupled gauge theory $\lambda \to \infty$
- Possible to move on further and actualy compute $a(1/\sqrt{\lambda} = 0, T)$ and $b(1/\sqrt{\lambda} = 0, T)$.
- Equate the potential with the free-energy from gravity.

$$2a\rho_{1,2}^2 + 2b\rho_{1,2}^4 + \log(1 - \rho_{1,2}) + f = -I_{1,2}$$

$$f = \log(2) - \frac{1}{2}$$

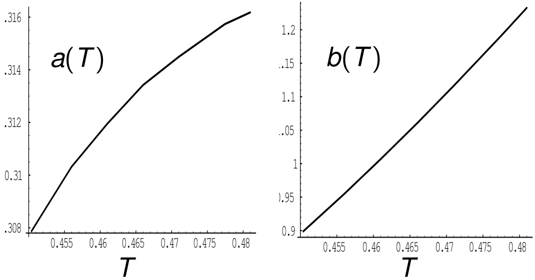
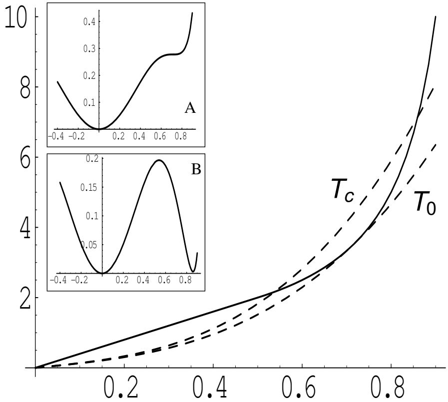
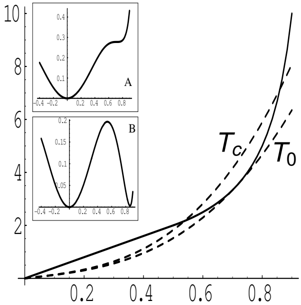


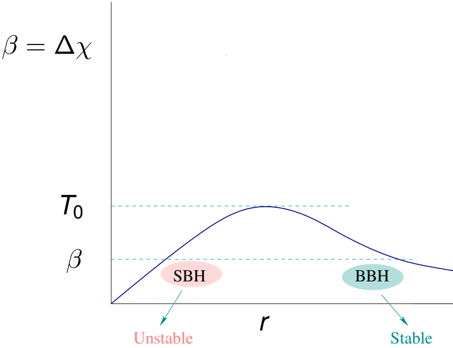
Figure: Plots of $a(T,0)$ and $b(T,0)$

Phases



Phases


A
B
T_c
T_0
0.2
0.4
0.6
0.8
2
4
6
8
10


$\beta = \Delta\chi$
T_0
β
SBH
BBH
r
Unstable
Stable

Finite λ corrections

- Relation between gauge coupling and $\alpha^{'}$

$$\frac{R^4}{\alpha^{'2}} = 4\pi g_{YM}^2 N = 4\pi\lambda$$

- Adding $\alpha^{'}$ corrections in the bulk gravity amounts to deviating from the $\lambda \to \infty$, thus bringing in $1/\sqrt{\lambda}$ corrections.
- Comparison with gravity

$$2a\rho_{1,2}^2 + 2b\rho_{1,2}^4 + \log(1-\rho_{1,2}) + f = -I_{1,2}$$

- We can now find the corresponding $1/\sqrt{\lambda}$ corrections to $a(0,T)$ and $b(0,T)$.

- Taylor expansion

$$a(T,1/\sqrt{\lambda}) = a(T,0) + \frac{1}{\sqrt{\lambda}}\frac{\partial a(T)}{\partial(1/\sqrt{\lambda})}\,|_{1/\sqrt{\lambda}=0} + \mathcal{O}(1/\lambda^{3/2})$$

$$b(T,1/\sqrt{\lambda}) = b(T,0) + \frac{1}{\sqrt{\lambda}}\frac{\partial b(T)}{\partial(1/\sqrt{\lambda})}\,|_{1/\sqrt{\lambda}=0} + \mathcal{O}(1/\lambda^{3/2})$$

- Compare Coefficients

$$2\frac{\partial a(T)}{\partial(1/\sqrt{\lambda})}\rho_{1,2}^2 + 2\frac{\partial b(T)}{\partial(1/\sqrt{\lambda})}\rho_{1,2}^4 = -\sqrt{\lambda}\delta I_{1,2}(T)$$

$$\begin{aligned}\delta I_{1,2}(T) &= \alpha^{'}\beta(\delta F_{1,2}) \\ &= -\frac{\beta}{\sqrt{2\lambda}}(3r_{1,2}^4 + 24r_{1,2}^2 + 9)\end{aligned}$$

- *b* decreases

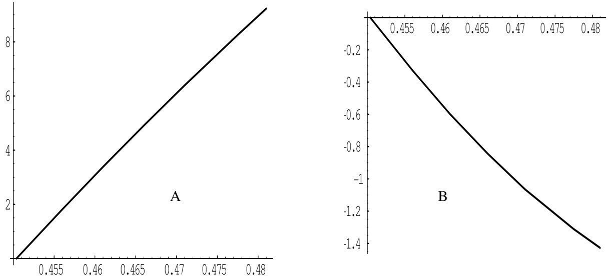
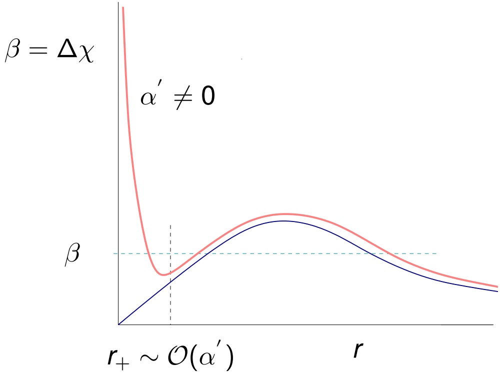


Figure: Plots of (A) $\partial a(T)/\partial(1/\sqrt{\lambda})$ and (B) $\partial b(T)/\partial(1/\sqrt{\lambda})$

- *b* may end up with a positive or negative sign in the weak coupling,

Modifying the matrix potential



- This comparison will however be valid as long as the radius of the small black hole is greater than α'.

Modified matrix potential

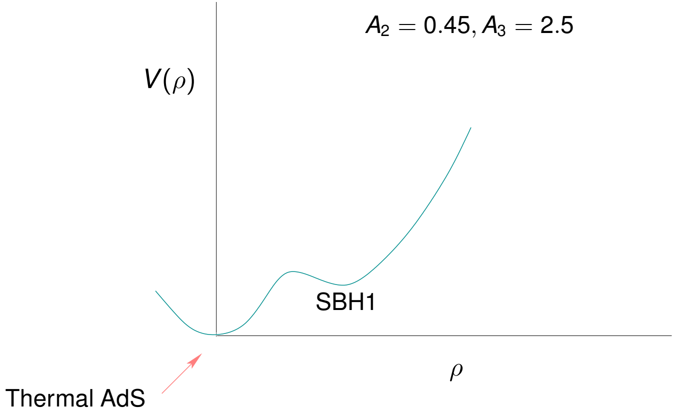
- Including higher order terms

$$S(\rho^2) = 2[A_4\rho^8 - A_3\rho^6 + A_2\rho^4 + (\frac{1-2A_1}{2})\rho^2]$$

- Saddle-point equations

$$\begin{aligned} \rho F(\rho) &= \rho \quad , \qquad 0 \leq \rho \leq \tfrac{1}{2}, \\ &= \tfrac{1}{4(1-\rho)} \quad , \quad \tfrac{1}{2} \leq \rho \leq 1 \end{aligned}$$

- Potential for various values of A_2 and A_3.



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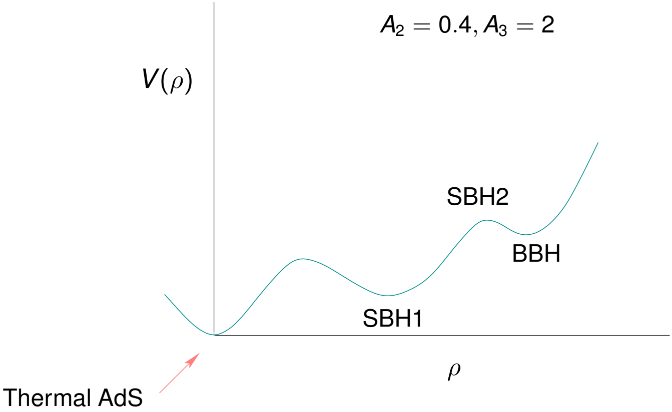

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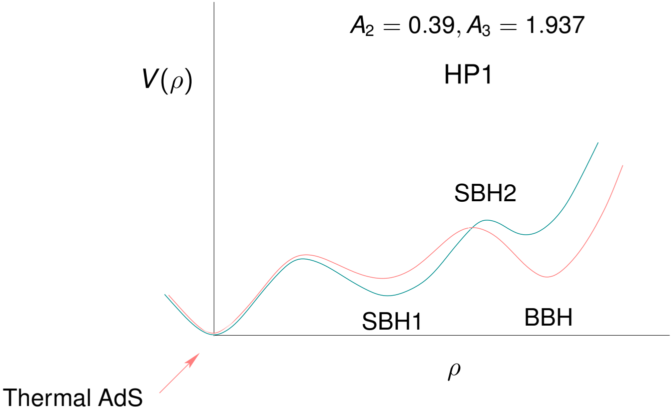

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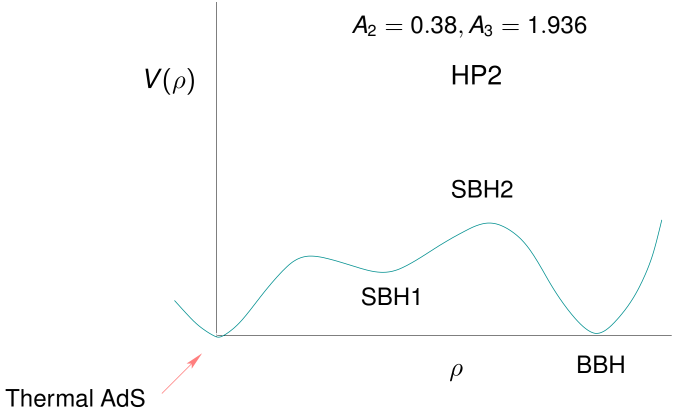

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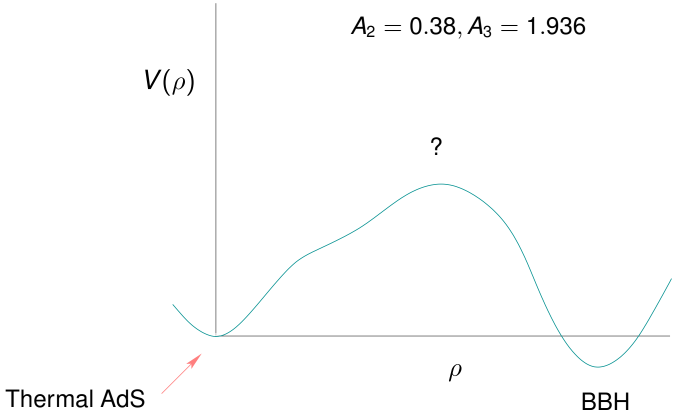


Witten's Kaluza-Klein Bubbles

- 1. Euclidean Kaluza-Klein vacuum

$$ds^2 = dx^2 + dy^2 + dz^2 + dt^2 + d\phi^2$$

- 2. Another solution with the same asymptotic form (Euclidean black hole)

$$ds^2 = \frac{dr^2}{1 - \alpha/r^2} + r^2 d\Omega^2 + \left(1 - \frac{\alpha}{r^2}\right) d\phi^2$$

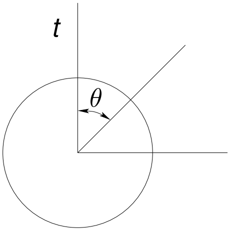
- The metric 2 is non-singular if the period of ϕ is $2\pi\sqrt{\alpha} = 2\pi R$

$$ds^2 = \frac{dr^2}{1 - (R/r)^2} + r^2 d\Omega^2 + \left(1 - \left(\frac{R}{r}\right)^2\right) d\phi^2$$

with $R < r < \infty$.

Kaluza-Klein Bubbles

- Analytic continuation to Minkowski space. Locate a plane that resembles $t = 0$.



$t \rightarrow it$ is equivalent to $\theta \rightarrow 1/2\pi + i\psi$

Kaluza-Klein Bubbles

- Metric with Minkowski signature

$$ds^2 = \frac{dr^2}{1-(R/r)^2} - r^2 d\psi^2 + r^2 \cosh^2\psi d\Omega^2 + \left(1 - \left(\frac{R}{r}\right)^2\right) d\phi^2$$

- Now since $R < r < \infty$ this metric defines a bubble expanding with time.
- KK vacuum decays into this bubble of NOTHING!

AdS Bubbles

- AdS Black hole

$$ds^2 = -(1 + \frac{r^2}{l^2} - \frac{m}{r^2})\, dt^2 + (1 + \frac{r^2}{l^2} - \frac{m}{r^2})^{-1} dr^2 + r^2 d\Omega^2$$

- Analytically continue the coordinates : $t \to i\chi, \theta \to \pi/2 + i\tau$.
- Bubble Metric :

$$ds^2 = V(r) d\chi^2 + \frac{dr^2}{V(r)} - r^2 d\tau^2 + r^2 \cosh^2\tau\, d\Omega^2$$

- Metric is non-singular if in the region $r \geq r_+$ if χ has a periodicity:

$$\Delta\chi = \frac{2\pi \bar{r} l (\bar{r}^2 + 2\bar{\alpha})}{(\bar{r}^2 + 2\frac{\bar{r}^4}{l^2})}$$

Bubble Solutions

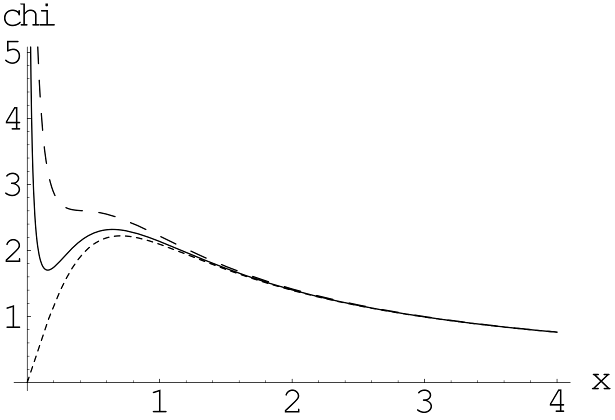


Figure: Plot of $\Delta\bar{\chi}$ as a function $x = \bar{r}$. The dashed line is for $\bar{\alpha} = 1/34$. The solid line is for $\bar{\alpha} = 1/50$ and the dotted line is for $\bar{\alpha} = 0$.

- Energy Momentum Tensor :

$$T^{\tau}_{\tau} = \frac{1}{\kappa_5 L^3} m$$

The bubble has lower energy for any value of r than the analytically continued AdS orbifold.

- Boundary : $dS^3 \times S^1$

$$ds^2 = d\chi^2 + L^2(-d\tau^2 + \cosh^2\tau d\Omega^2_{n-2})$$

$$L = \sqrt{2\hat{\alpha}}\left[1 - \left(1 - \frac{4\hat{\alpha}}{l^2}\right)^{\frac{1}{2}}\right]^{-\frac{1}{2}}$$

$$
\begin{tikzcd}
 & \text{Bulk} & \text{Boundary} & \\
 & AdS_5 \arrow[dd] & S^4 \arrow[dd] & \\
\text{An. C AdS Orbifold} & & & S^1 \times dS^3 \\
 & X \arrow[ul] \arrow[dl] & M \arrow[ur] & \\
\text{Bubble} & & &
\end{tikzcd}
$$

Summary

- Phases in the bulk gravity and on the boundary gauge theories in the presence of higher derivetive (Gauss-Bonnet term).
- Phase structure can change once higher derivative terms are added.
- Proposal for a matrix model that captures the phases on the boundary
- Decay into bubbles of nothing (only a gravity analysis)