Spacetime superpotentials for B-branes in LG models

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Motivation

A nice intermediate step to obtaining the standard model from string theory is to look for $\mathcal{N} = 1$ compactifications of string theory in four dimensions.
The effective field theory at low energy is specified by the following functions

- The Kähler potential, $K(\phi, \bar{\phi})$,
- The superpotential $\mathcal{W}(\phi)$, and
- The (complexified) gauge coupling constants, $f(\phi)$.

The last two objects are holomorphic (in the chiral superfields) and can be computed in topological string theory.

Motivation

- Superpotentials can arise from fluxes through compact cycles being switched on – usually these are computed using the Gukov-Vafa-Witten formula.
- They can also arise from the worldvolume theory of branes that may be added to cancel tadpoles – say in orientifold theories. This is sometimes called the brane superpotential.
- The superpotential $\mathcal{W}$ has been computed for non-compact examples. Is there a systematic method to compute it in compact examples?

Motivation

- For type II compactifications with $\mathcal{N} = 2$ supersymmetry, mirror symmetry has proved useful in summing up non-perturbative contributions coming from worldsheet instantons (Gromov-Witten; Gopakumar-Vafa).
- One important ingredient in mirror symmetry is the closed-string mirror map. This is a highly non-trivial change of variables.
- An important ingredient in this computation is the observation of Candelas et. al. that the change of variables is given by a solution of a Picard-Fuchs differential equation.
- Is there an analogue for open-strings? Yes, for some non-compact examples (Mayr;SG-Jayarman-Sarkar; Mayr-Lerche-Warner). Is there a diff. eqn. for compact examples as well? (Walcher)

Matrix Factorizations and Superpotentials

- A matrix factorization of a function $W(z)$ is given by two $N \times N$ matrices $F(z)$ and $G(z)$ satisfying

$$F(z) \cdot G(z) = G(z) \cdot F(z) = W(z) \mathbf{1}_{N \times N} \ .$$

- D-branes in Landau-Ginzburg models can be related to matrix factorizations (Kapustin-Li, Brunner-Herbst-Lerche-Scheuner)
- The open-string spectrum is given by the cohomology of a BRST operator Q constructed from F and G.
- Open and closed string deformations can obstruct (spoil) matrix factorizations. (Ashok-Diaconescu-Dell' Acqua, Hori-Walcher)
- Such obstructions can be encoded in an effective superpotential, $\mathcal{W}$. Direct computation can be hard beyond simple examples.

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- D-branes in Landau-Ginzburg models can be related to matrix factorizations (Kapustin-Li, Brunner-Herbst-Lerche-Scheuner)
- Some matrix factorizations can also be represented by simple boundary conditions in LG models with boundary. (SG-Jayaraman-Sarkar; Ezhuthachan-SG-Jayaraman)
- We will discuss a method to compute $\mathcal{W}$ in this setting.

Why are LG models useful?

- They flow to non-trivial CFT's in the infrared (IR).
- One has a better handle on perturbations that appear in the superpotential such as complex structure moduli.
- Some computations are like in free-field theory.

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Intrepretation of results can be tricky and need careful analysis.

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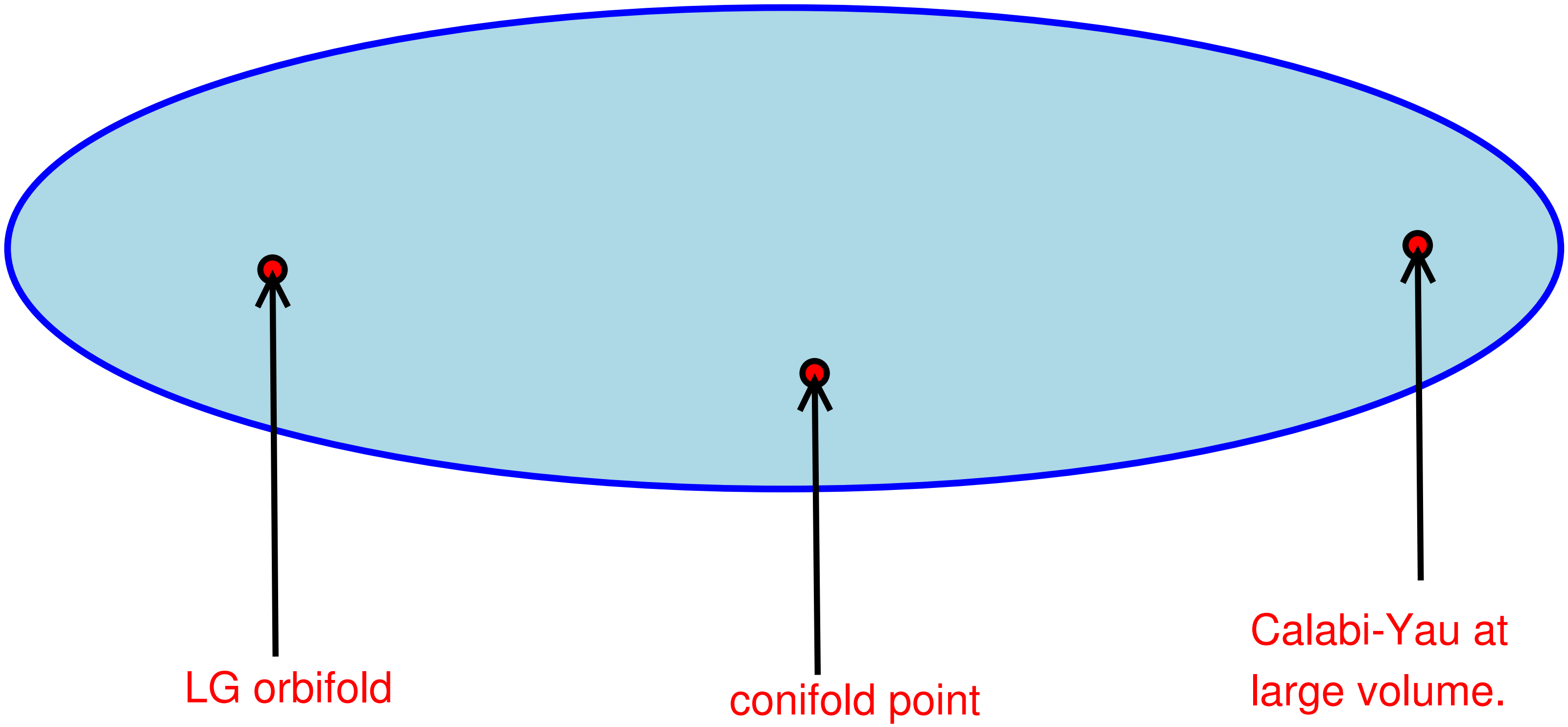
- They flow to non-trivial CFT's in the infrared (IR).
- One has a better handle on perturbations that appear in the superpotential such as complex structure moduli.
- Some computations are like in free-field theory.
- The main motivation is that the computation for a Calabi-Yau threefold is not much different from that of a minimal model.

Kahler moduli space of the quintic



The topological B-model is independent of Kähler moduli at tree level.

Plan of talk

- Motivation
- A quick introduction to topological LG models
- Computing the superpotential – I: the A-minimal models
- Computing the superpotential – II: the cubic torus
- Concluding remarks

LG models

- Two-dimensional LG models with $(2,2)$ supersymmetry are constructed from chiral and anti-chiral superfields.
- Chiral superfields have the following expansion $(\alpha = \pm)$

$$\Phi = \phi + \sqrt{2}\theta^\alpha \psi_\alpha + \theta^\alpha \theta_\alpha F$$

- The most general renormalizable action for such a theory has an action

$$\begin{aligned} S &= S_K + S_W \\ &= \int d^2x \left(\int d^4\theta K(\Phi, \bar{\Phi}) - \lambda \int d^2\theta \, W(\Phi) - \bar{\lambda} \int d^2\bar{\theta} \, \bar{W}(\bar{\Phi}) \right) \end{aligned}$$

- We will find it useful to define the following combinations: $\tau = (\psi_+ - \psi_-)/\sqrt{2}$ and $\xi = (\psi_+ + \psi_-)/\sqrt{2}$.

LG models

- We will assume that the superpotential W is quasi-homogeneous

$$W(\lambda^{\alpha_i/2}\Phi_i) = \lambda\, W(\Phi_i)\ .$$

- There is a lot of evidence that such LG models flow in the IR to CFT's with central charge $\hat{c} = \sum_i (1-\alpha_i)$.
- In models with several fields, we will be interested in LG orbifolds with projections onto states with (half-)integral R-charge.
- Example 1: It involves a single chiral superfield with $W = \phi^{k+2}/(k+2)$ with $\hat{c} = k/(k+2)$ – the relevant CFT is the A-minimal model.

LG models

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- In models with several fields, we will be interested in LG orbifolds with projections onto states with (half-)integral R-charge.
- Example 2: It involves three chiral superfields and a cubic superpotential $W = c_{ijk}\phi^i\phi^j\phi^k$ and a $\mathbb{Z}_3$ orbifolding. The CFT has $\hat{c} = 1$ and is the 1^3 Gepner model.

Topological LG models

- These are topologically twisted versions of the $(2, 2)$ models.
- We will consider the topological B-twist. This has two BRST charges which we will denote by $\mathcal{Q}_{\pm}$.
- In LG models with boundary, we will assume that one linear combination, $\mathcal{Q}$, is preserved by the boundary conditions.
- Observables in the topological model are given by the cohomology of $\mathcal{Q}$.
- In the action, only the holomorphic part of S_W is non-trivial. S_K for instance, is $\mathcal{Q}$-exact. Thus, the topological partition function is independent of the Kähler potential and depends holomorphically on the parameters(moduli) in W.

The topological partition function

- The topological partition function is formally defined by the following path-integral:

$$Z_{top} \equiv \int_{disk} [d\Phi] e^{-S_K - S_W} \; P\left(e^{-S_\partial}\right)$$

  with S_∂ representing the boundary perturbations.

- We will treat S_W and S_∂ perturbatively. Let $\langle\!\langle \cdots \rangle\!\rangle$ denote correlation functions in the free-theory i.e., with $W = 0$. Then,

$$Z_{top} = \sum_{m,n=0}^{\infty} \left\langle\!\!\left\langle \frac{1}{n!} \left(S_W\right)^n \frac{1}{m!} P\left(S_\partial\right)^m \right\rangle\!\!\right\rangle ,$$

  is formally equivalent to the the path-integral.

$$\mathcal{W} = Z_{top}$$

- It is known that the open-superstring partition function on a disk gives the open-string field theory action. (Witten; Shatashvili, Kutasov-Martiniec-Moore; Niarchos-Prezas).
- This has been used by Kutasov, Marino and Moore to compute the *exact* action for the tachyons in order to verify Sen's conjectures on tachyon condensation. (see also: Gerasimov and Shatashvili)
- For $\mathcal{N} = 1$ supersymmetric compactifications, Z_{top} can be identified with the the brane superpotential, $\mathcal{W}$.
- For non-geometric examples, Z_{top} can be identified with the obstruction superpotential – these encode higher order obstructions to marginality or obstructions to the existence of matrix factorizations.

Our goal

Compute $\mathcal{W} \equiv Z_{top}$ in the topological LG model as a function of both closed string parameters and open-string deformations.

Such a computation was first done for the quintic(!), where the first correction from closed-string moduli was computed.

(Douglas, SG, Jayaraman & Tomasiello)

As we will see, the surprise is that the computation of $\mathcal{W}$ can be carried out to all orders.

Example 1: The A-minimal model

The A_{k+1}-minimal model

- The superpotential for the A_{k+1}-minimal model is

$$W_0 = \frac{\phi^{k+2}}{k+2} \equiv g_0 \ \phi^{k+2}$$

- The central charge is $\hat{c} = \frac{k}{k+2}$.
- The $U(1)_R$-assignments are fixed by requiring that W_0 have scaling dimension 2

ϕ	(τ, ξ)	$(\bar{\tau}, \bar{\xi})$	W_0
$\frac{2}{k+2}$	$\frac{-k}{k+2}$	$\frac{k}{k+2}$	2

- The (closed string) BRST observables form the chiral ring $\mathbb{C}[\phi]/dW_0$. Explicitly, the elements of the ring are the chiral primaries: 1, ϕ, . . . , ϕ^k.

Bulk perturbations

- The CFT can be perturbed by adding relevant perturbations given by elements of the chiral ring. In the LG model, this corresponds to deforming W_0.

$$W = W_0 - \sum_{j=2}^{k+2} g_j(t)\ \phi^{k+2-j}\ ,$$

where t_j are flat-coordinates and $g_j(t) = t_j + \cdots$.

- For instance,

$$W = \frac{\phi^5}{5} - t_2\phi^3 - t_3\phi^2 - (t_4 - t_2^2)\phi - (t_5 - t_2 t_3)\ .$$

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- [DVV] In terms of flat coordinates, the three-point function is given by the third derivative (w.r.t. t_i) of the topological partition function, $F(t_i)$

$$c_{ijk}(t) \equiv \langle \mathcal{O}_i \mathcal{O}_j \mathcal{O}_k \rangle = \partial_i \partial_j \partial_k F(t)\ ,$$

where $\mathcal{O}_j = -\partial_j W = \phi^{k+2-j} + \cdots$.

Adding a boundary

- Take the topology of the worldsheet to be the upper half plane: $x \in (-\infty, +\infty)$ and $y \geq 0$. The boundary is the x-axis.
- We need to specify boundary conditions in the LG model. We will consider boundary conditions that preserve half of the $(2, 2)$ supersymmetry and is compatible with the topological B-twist.
- We choose the Dirichlet boundary condition $\phi = 0$ and $\tau = 0$.
- Naively, one cannot impose a Neumann boundary condition due to the presence of the Warner term.
- This is to be identified with the boundary state $|L = 0\rangle_B$ in the CFT.

Boundary perturbations

- The only Q-closed boundary operator one can construct is $\bar{\xi}$.
- One can turn on boundary perturbations using this operator – it has R-charge $\frac{k}{k+2} < 1$ and is a relevant perturbation.
- We will also need to consider integrated operators. One has (X is the boundary coupling constant)

$$\begin{array}{cccc} & \Psi^{(0)} = X\ \bar{\xi} & , & \Psi^{(1)} = X\ \partial_y \bar{\phi} \\ R\text{–charge}: & \dfrac{k}{k+2} & , & \dfrac{-2}{k+2} \end{array}$$

$S_\partial = X \int dx \Psi^{(1)}$ is the perturbation added to the action.

Boundary perturbations

- The only $\mathcal{Q}$-closed boundary operator one can construct is $\bar{\xi}$.
- One can turn on boundary perturbations using this operator – it has R-charge $\frac{k}{k+2} < 1$ and is a relevant perturbation.
- Similarly, for the bulk deformations, one has

$$\mathcal{O}_f^{(0)} = f_d(\phi) \qquad , \qquad \mathcal{O}_f^{(2)} = -\frac{\partial^2 f_d}{\partial \phi^2} \tau \xi$$

$$R\text{−charge} : \quad \frac{2d}{k+2} \qquad , \qquad \left(\frac{2d}{k+2} - 2 \right)$$

where $f_d(\phi)$ is a function of ϕ of degree $d \leq k$.

The topological partition function

- The topological partition function is formally defined by the following path-integral:

$$\mathcal{W} \equiv \int_{disk} [d\Phi] e^{-S_K - S_W} \; P\left(e^{-S_\partial}\right) \; .$$

- We will treat S_W and S_∂ perturbatively. Let $\langle\!\langle \cdots \rangle\!\rangle$ denote correlation functions in the free-theory i.e., with $W = 0$. Then,

$$\mathcal{W} = \sum_{m,n=0}^{\infty} \left\langle\!\!\left\langle \frac{1}{n!} (S_W)^n \, \frac{1}{m!} P\,(S_\partial)^m \right\rangle\!\!\right\rangle \, ,$$

is formally equivalent to the the path-integral.

Issues to consider

- Need to fix $SL(2,\mathbb{R})$ invariance We fix this by choosing one bulk operator as a zero-form located at the point (x_0, y_0) and one boundary operator as a zero-form located at $x = +\infty$. All other operators are chosen to be integrated ones.
- R-charge selection rule The only non-vanishing correlators $\langle\!\langle \cdots \rangle\!\rangle$ occur when the sum of the R-charges of all operators equals $\hat{c} = k/(k+2)$.
- Fermion zero-modes There is one fermion zero-mode coming from $\bar{\xi}$. The bulk topological theory has one more zero-mode which is removed by the boundary condition.
- All the fields must be contracted with some other field.

The computation

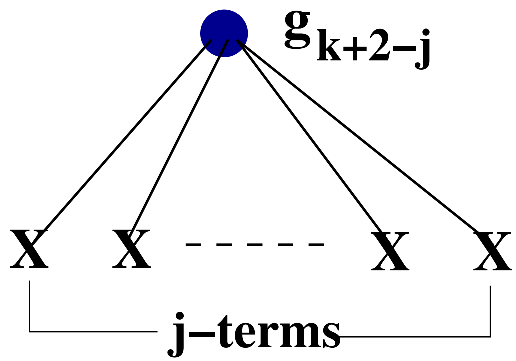
- For the A-minimal model, one finds that the only non-vanishing contribution occurs when there is one bulk insertion.
- Recall that the LG superpotential including bulk deformations is

$$W = \sum_{j=0, j\neq 1}^{k+2} g_j(t)\ \phi^{k+2-j} = \sum_{j=0, j\neq k+1}^{k+2} g_{k+2-j}(t)\ \phi^j\ .$$

- For one bulk operator $(\phi)^j$, the R-charge constraint requires $(j+1)$ boundary operators.

$$\frac{\partial^2 \mathcal{W}}{\partial\lambda\partial X} = \sum_{j=0}^{k+2} \frac{g_{k+2-j}(t) X^j}{j!} \left\langle\!\!\left\langle \phi^j(x_0, y_0)\ P \left(\prod_{i=1}^{j} \int dx_i \partial_y \bar{\phi}\right) \bar{\xi}(+\infty) \right\rangle\!\!\right\rangle$$

The computation



- Above, each line indicates the bulk-boundary contraction between $\phi(x_0, y_0)$ and $\partial_y \bar{\phi}(x_i)$ given by the free-field propagator (for a Dirichlet b.c.)

$$\langle \phi(x_0, y_0) i\partial_y \bar{\phi}(x_i) \rangle = \frac{1}{\pi} \frac{y_0}{(x_0 - x_i)^2 + y_0^2} \ .$$

- The integrals are easy to do and each is normalised to give 1.
- The total number of possible contractions equals $j!$.

The result

Combining all this we get (after integrating),

$$\mathcal{W} = \sum_{j=0}^{k+2} g_{k+2-j}(t)\ \frac{X^{j+1}}{j+1}$$

- The above result can be extended to the include the $L \neq 0$ boundary states that appear in CFT.
- For instance, the $L = 1$ boundary state is the bound state of two $L = 0$ states.
- This is incorporated in the LG model using the Chan-Paton trick. Making X into a 2×2 matrix with arbitrary entries is equivalent to two $L = 0$ boundaries.
- Then the above result can be carried over with the replacement: $X^{j+1} \to \mathrm{Tr}(X^{j+1})$.

The HLL conjecture

So we finally obtain a very simple result that is valid for all bound states of the $L = 0$ boundary state in CFT. This is known to give all the CFT boundary states.

$$\mathcal{W}(t, X) = \sum_{j=0}^{k+2} g_{k+2-j}(t) \, \frac{\mathrm{Tr}(X)^{j+1}}{j+1}$$

This result was conjectured by Herbst, Lazaroiu and Lerche in hep-th/0402110. It was obtained (algebraically and experimentally) by imposing A_∞-constraints, bulk-boundary crossing symmetry and the Cardy (sewing) constraint on the TFT correlators for specific minimal models at low values of k.

This is a proof of the HLL conjecture.

Example 2: The cubic torus

The cubic torus

- The LG description involves three chiral superfields Φ^i with a superpotential

$$W = c_{ijk}\phi^i\phi^j\phi^k = g_0\left((\phi^1)^3 + (\phi^2)^3 + (\phi^3)^3\right) + g_1\phi^1\phi^2\phi^3$$

- There is an orbifold action:

$$\phi^i \to \omega\phi^i \quad \text{where} \quad \omega = e^{2\pi i/3}\ .$$

  So one is dealing with an orbifold of a LG model.

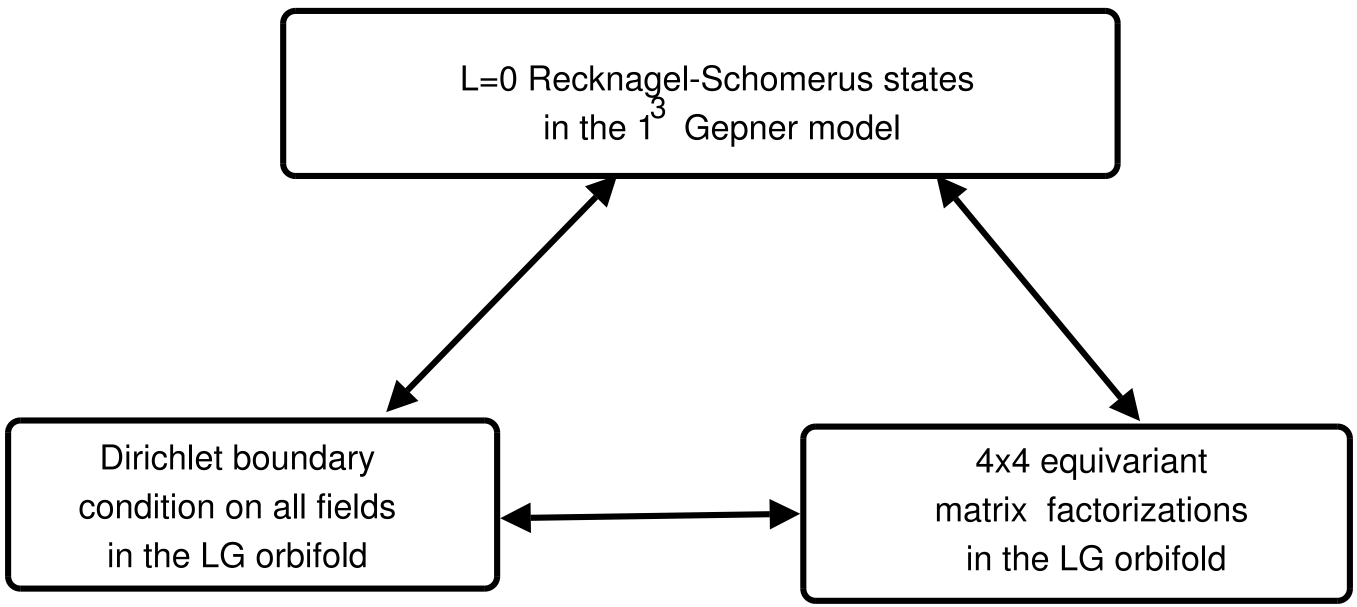
- The IR fixed point of the LG model is the 1^3 Gepner model.
- Without the superpotential, one has a $\mathbb{C}^3/\mathbb{Z}_3$ orbifold. This is the analogue of the free theory in the previous example.

The cubic torus

- Geometrically, the torus $\mathcal{T}$ is given by the hypersurface $W = 0$ in $\mathbb{P}^2$ with complex structure modulus (and flat coordinate) τ implicitly given by $\frac{g_1}{g_0} = -3a(\tau)$.
- The relationship between a and τ is given through the j-function
$$j(\tau) = \left(\frac{3\,a\,(a^3+8)}{a^3-1}\right)^3 .$$
Note that a fixed value of τ gives 12 values of a.
- In the differential equation for the periods, one needs to set
$$(g_0)^{-1} = \sqrt{\frac{1-a^3(\tau)}{3\,a'(\tau)}} = \frac{1}{3\sqrt{2\pi i}}\,\frac{\eta(\tau)}{\eta^3(3\tau)}$$
- The mirror torus $\hat{\mathcal{T}}$ is one with complex structure $\hat{\tau} = e^{2\pi i/3}$ and (complexified) Kähler modulus $\hat{\rho} = \tau$.

Adding a boundary

We shall focus on the situation where one imposes Dirichlet boundary conditions on all fields: $\phi^i = \tau^i = 0$. In the Gepner model, this boundary condition gets mapped to the $L_i = 0$ Recknagel-Schomerus states.



Boundary deformations

- On the boundary now there are several $\mathcal{Q}$-closed operators. One can consider $\bar{\xi}_i$ (R-charge $1/3$) – this is similar to the minimal model.
- There are more – $\bar{\xi}_i\bar{\xi}_j$ and $\bar{\xi}_1\bar{\xi}_2\bar{\xi}_3$ with R-charges $2/3$ and 1, respectively.
- We will focus on two boundary perturbations:

$$\Psi^{(0)} = X^i\ \bar{\xi}_i \quad , \quad \Psi^{(1)} = X^i\ \partial_y\bar{\phi}_i$$
$$\Omega^{(0)} = U\epsilon^{ijk}\bar{\xi}_i\bar{\xi}_j\bar{\xi}_k \quad , \quad \Omega^{(1)} = 3U\epsilon^{ijk}\bar{\xi}_i\bar{\xi}_j\partial_y\bar{\phi}_k$$

- Using the R-charge assignments, we see that the X-perturbation is a relevant one while the Ω-perturbation is a marginal one.

Boundary deformations

- However, since the free theory is an orbifold, there are three boundary states corresponding to fractional zero-branes.
- This is incorporated by using Chan-Paton factors which take into account the spectrum of open-strings connecting the various fractional zero-branes.

$$X^i = \begin{pmatrix} 0 & x^i_{12} & 0 \\ 0 & 0 & x^i_{23} \\ x^i_{31} & 0 & 0 \end{pmatrix} \quad , \quad U = \begin{pmatrix} u_1 & 0 & 0 \\ 0 & u_2 & 0 \\ 0 & 0 & u_3 \end{pmatrix} .$$

- Thus the X^i are boundary condition changing operators while U is a boundary condition preserving operator.

4×4 matrix factorizations

- As already mentioned, these b.c.'s are related to matrix factorizations where F and G are 4×4 matrices with W being the cubic superpotential of the LG model.
- The open-string spectrum for this matrix factorization matches the one we discussed. (SG-Jockers-Lerche-Warner; Hori-Walcher)
- This is not always true. There can be extra deformations.
- The marginal deformation given by Ω deforms the matrix factorization to a 3×3 matrix factorization. (Brunner-Herbst-Lerche-Scheuner; SG-Jockers-Lerche-Warner)

The topological partition function

- It is useful to separate the partition function by the number of bulk insertions: $\mathcal{W} = \sum_{n=0}^{\infty} \mathcal{W}_n$.
- R-charge considerations imply that $\mathcal{W}_n$ (for $n \neq 0$) equals

$$\left\langle\!\!\left\langle \frac{1}{n!} V_W^{(0)} \left(\int V_W^{(2)} \right)^{n-1} \frac{P}{n!3!} \left[\left(\int \Omega^{(1)} \right)^n \left(\int \Psi^{(1)} \right)^3 \right] \Omega^{(0)}(\infty) \right\rangle\!\!\right\rangle$$

- When there are no bulk insertions, one has

$$\begin{aligned} \mathcal{W}_0 &= \left\langle\!\!\left\langle \Psi^{(0)}(0)\Psi^{(0)}(1)\Psi^{(0)}(\infty) \right\rangle\!\!\right\rangle = \mathrm{Tr}(X^i X^j X^k) \left\langle\!\!\left\langle \bar{\xi}_i \bar{\xi}_j \bar{\xi}_k \right\rangle\!\!\right\rangle \\ &= \epsilon_{ijk} \mathrm{Tr}(X^i X^j X^k) \end{aligned}$$

This is known to be the $\mathbb{C}^3/\mathbb{Z}_3$ superpotential.

The systematics

The $\mathcal{W}_n$ can be written as

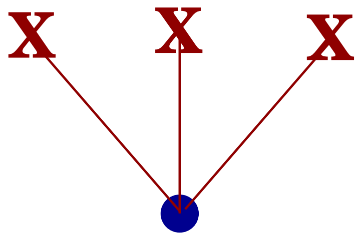
$$\mathcal{W}_n = \mathcal{I}_n \, \mathcal{C}_n \, g_0^n (u_1 + u_2 + u_3)^n$$

where

- $\mathcal{I}_n$ includes the contribution from the integrals,
- $\mathcal{C}_n$ contains the contractions of the n copies of the $\square\!\square\!\square$ tensor of $SU(3)$, c_{ijk} with the boundary X's and the antisymmetric tensor ϵ^{ijk} from the Ω's.
- One can show that only the combination $u \equiv (u_1 + u_2 + u_3)$ appears.

The integrals simplify in the limit when we take the bulk zero-form operator close to the boundary.

Some details



$$\mathcal{C}_1 = 3c_{ijk}\mathrm{Tr}\left(X^iX^jX^k\right)$$

$$\mathcal{W}_1 = 3\mathcal{I}_0\left(3\kappa_{111} - \frac{3}{2}a\big(\kappa_{123} + \kappa_{132}\big)\right)g_0u$$

where we define the following useful combinations:

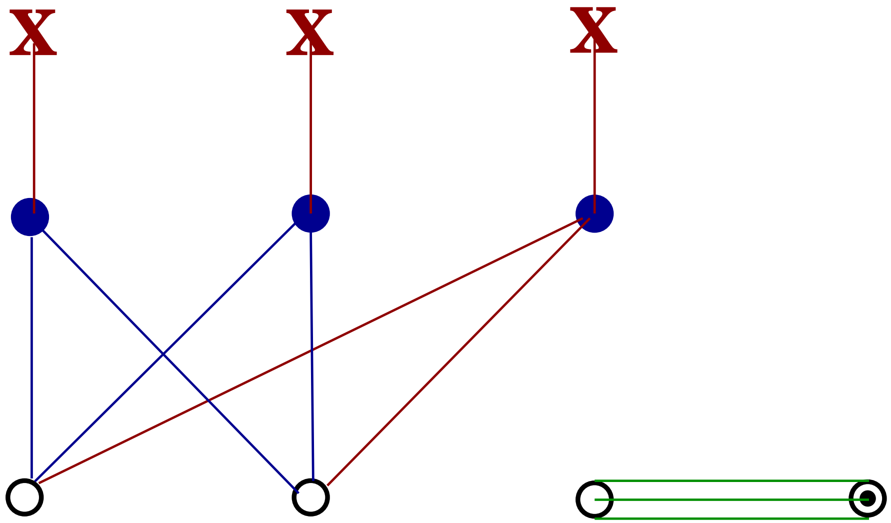
$$\kappa_{111} = \tfrac{1}{3}\textstyle\sum_i \mathrm{Tr}\left(X^iX^iX^i\right) \ ,$$

$$\kappa_{123} = \mathrm{Tr}\left(X^1X^2X^3\right) \ ,$$

$$\kappa_{132} = \mathrm{Tr}\left(X^1X^3X^2\right) \ .$$

Some details

$\mathcal{C}_2$ vanishes. So the next non-vanishing term is given by the third-order term.



$$\mathcal{W}_3 = \mathcal{I}_3 \left(-\frac{9}{2} a^2 \kappa_{111} + \left(3 - \frac{3}{4} a^3 \right) (\kappa_{123} + \kappa_{132}) \right) (g_0 u)^3 \ ,$$

General result $(\kappa_{123} - \kappa_{132})$ appears only in $\mathcal{W}_{2n}$ while $(\kappa_{123} + \kappa_{132})$ and κ_{111} appear in $\mathcal{W}_{2n+1}$ alone.

Summing up

$$\mathcal{W}_B = \Delta^B_{111}(\tau, g_0 u)\kappa_{111}(X) + \Delta^B_{123}(\tau, g_0 u)\kappa_{123}(X) + \Delta^B_{132}(\tau, g_0 u)\kappa_{132}$$

From the properties of the $\mathcal{W}_n$, we find

$$\begin{aligned} \Delta^B_{111}(\tau, -g_0 u) &= -\Delta^B_{111}(\tau, -g_0 u) \\ \Delta^B_{123}(\tau, -g_0 u) &= -\Delta^B_{132}(\tau, -g_0 u) \end{aligned}$$

We will see that these properties are compatible with the computation done on the mirror torus.

The Δ^B can be viewed as an open-string three-point function

deformed by the bulk and boundary modulus, τ and u.

Verifying the results

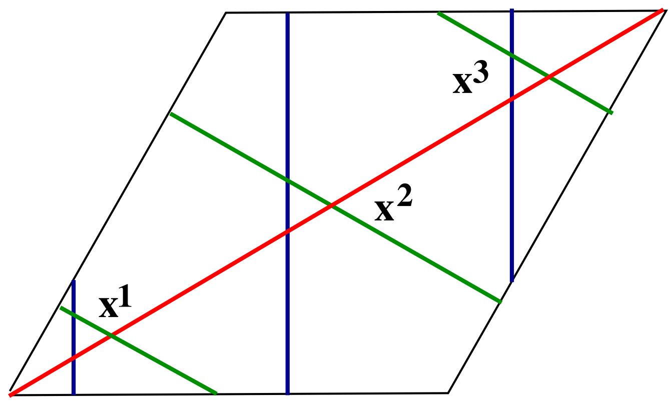
Under mirror symmetry, the topological B-model gets mapped to the topological A-model on the mirror.

Superpotentials are classical objects in the B-model while they are quantum objects in the A-model. All contributions arise from worldsheet instantons. (Kachru-Katz-Lawrence-McGreevy)

The simplicity of our example enables us to easily write out the disk instanton contributions and we use it as a check of our computation.

The mirror theory

Under the mirror transform, B-branes get mapped to A-branes. The three branes that we considered thus get mapped to branes that are special Lagrangian one-cycles on the mirror torus $\hat{\mathcal{T}}$.



The boundary changing operators (x^i_{ab}) are operators located at the intersection points – there are nine of them. The boundary moduli, u_i are the positions of the three branes.

The A-model result

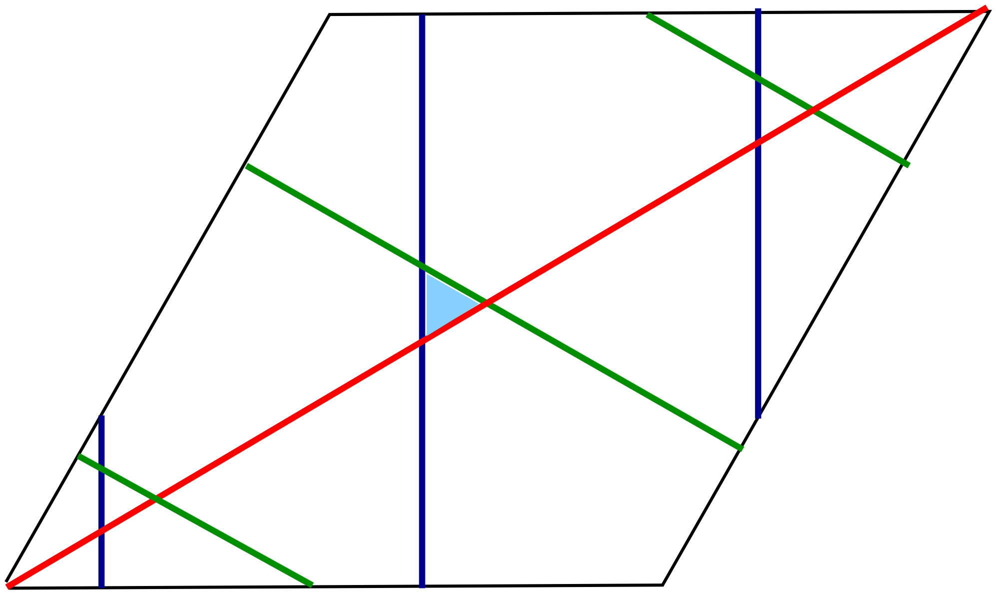
The three-point functions in the topological A-model vanish perturbatively and get contributions only from worldsheet instantons – these are disk instantons.
Schematically, one finds

$$\Delta_{ijk} \sim \sum_l e^{2\pi A^{(l)}_{ijk}(\hat{\beta})} e^{2\pi i W^{(l)}_{ijk}(\hat{\alpha})}$$

where $\hat{u} = \sum_i \hat{u}_i = \hat{\alpha} + \hat{\rho}\beta$ is the position modulus, $A^{(l)}_{ijk}$ is the area of the disk instanton and $W^{(l)}_{ijk}$ is the Wilson line contribution.

This result has been computed by Polishchuk-Zaslow; Cremades-Ibanez-Marchesano. We quote their result as adapted by Brunner, Herbst, Lerche and Walcher.

Disk Instantons in the A-model

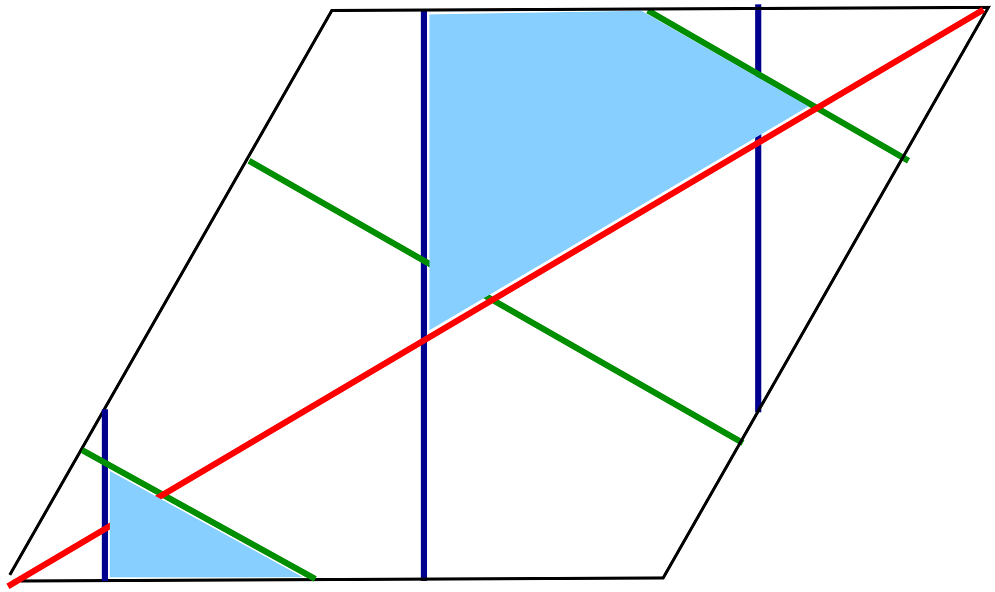


A contribution to Δ_{111}

$$\Delta_{111} = \sum_{m\in\mathbb{Z}} q^{\frac{3}{2}\left(m-\frac{1}{2}\right)^2} e^{2\pi i\left(m-\frac{1}{2}\right)\left(u_A-\frac{1}{2}\right)} \quad .$$

These are θ-functions of characteristic three.

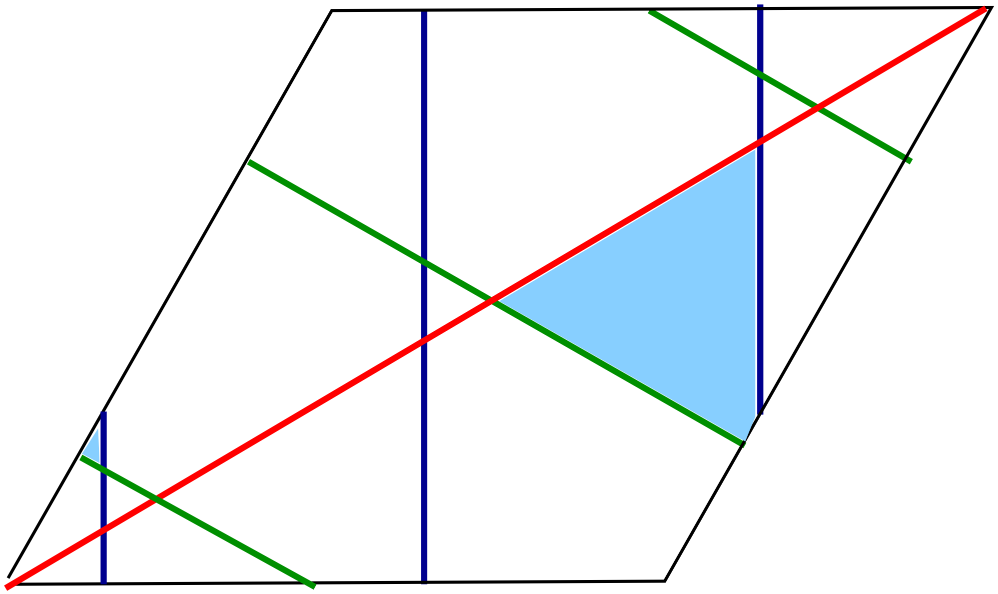
Disk Instantons in the A-model



A contribution to Δ_{123}

$$\Delta_{123} = e^{\frac{2}{3}i\pi} \sum_{m\in\mathbb{Z}} q^{\frac{3}{2}\left(-\frac{1}{3}+m-\frac{1}{2}\right)^2} e^{2\pi i\left(-\frac{1}{3}+m-\frac{1}{2}\right)\left(u_A-\frac{1}{2}\right)} \quad .$$

Disk Instantons in the A-model



A contribution to Δ_{132}

$$\Delta_{132} = e^{\frac{-2}{3}i\pi} \sum_{m\in\mathbb{Z}} q^{\frac{3}{2}\left(-\frac{2}{3}+m-\frac{1}{2}\right)^2} e^{2\pi i\left(-\frac{2}{3}+m-\frac{1}{2}\right)\left(u_A-\frac{1}{2}\right)} \quad .$$

Finding the open-string mirror map

- We need to figure out the change of variables that is needed to match our computation to the A-model result.
- We make the following ansatz

$$\begin{aligned} u_A &= \mathcal{N}_u(\tau)u_B + u_0(\tau) \\ X_A &= \mathcal{N}_X(\tau, u_B)X_B \end{aligned}$$

- The normalizations depend on the closed-string modulus τ.
- The additivity of the u's implies that the change of variable from u_A to u_B must be linear.
- R-charge considerations imply that X_A must be proportional to X_B – however, its normalization can depend on u_B as well.

Figuring out the normalization

- We match the two results by requiring

$$\mathcal{N}_X^3\ \mathcal{W}_A(\tau, \mathcal{N}_u u_B + u_0, X_B) = \mathcal{W}_B(\tau, g_0 u_B)$$

- The choice of g_0 from the diff. eqn. for periods makes $\mathcal{N}_u$, a τ independent constant. This implies that u_B transforms like a point on the torus.

$$\mathcal{N}_X = \frac{3i\mathcal{I}_0}{\eta(\tau)} \exp\left(2G_2(\tau)\mathcal{N}_u^2 u^2/3\right)\ f(\tau, u^2)\ .$$

where $G_{2k} = \sum'_{m,n\in\mathbb{Z}}(m\tau+n)^{-2k}$ is the Eisenstein series of weight $2k$. $f(\tau, u^2) = 1 + \mathcal{O}(u^4)$ is a modular invariant function.

Remarks

- The open-string mirror map is highly overdetermined and hence its very existence is a non-trivial check of the perturbative treatment.
- From a practical viewpoint, we have checked terms to several orders.
- We have also made use of the modular properties to have additional checks.
- The Eisenstein series $G_2(\tau)$ is actually not a modular form – $\hat{G}_2 = G_2 - \pi/\mathrm{Im}(\tau)$ has nice modular transformation properties but is not holomorphic. Holomorphic anomaly?
- Using matrix factorizations, Brunner, Herbst, Lerche and Walcher have also computed the three-point function which matches our results.

Conclusion

- The computation works for Calabi-Yau threefolds though modular properties are not well understood. (see recent paper by Aganagic *et al.*.)
- Walcher has recently obtained differential equation in the B-model which matches the disk instantons for a special class of disk instantons!
- Is there a differential equation satisfied by $\mathcal{N}_X$?
- The theta functions satisfy the heat equation – can one derive this from first principles for the superpotential that we computed?
- Need to extend this method to include short branes on the torus.

THANK YOU