Holography on the Horizon and at Infinity

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Reference:

Phys.Rev.D74:044007,2006. (with Rajesh Gopakumar)

Work in progress (with D. Astefanesei and R. Gopakumar)

Introduction

- Holographic Hypothesis tells us
  - Gravity in Bulk $\leftrightarrow$ Theory or information on the Boundary.
- For a Black Hole spacetime there are two boundaries,
  - Inner boundary or horizon
  - The asymptotic infinity
- Apparently there are two different holographic descriptions of the blackhole. One in terms of the horizon and the other in terms of the boundary theory.

- To gain a better understanding of this holographic description in theories of gravity we restrict our discussions only to the entropy of the black holes.

- Question
  - What are the Holographic ways to calculate BH entropies in AdS space?

- There are two Methods
  - Noetherian Method (Wald Approach) - Entropy is given in terms of local quantities evaluated on the horizon.
  - By AdS/CFT - the entropy of a black hole in AdS space is related to the thermodynamic entropy of the boundary gauge theory at a finite temperature.
- The free energy (or entropy) of the Euclidean thermal field theory is the same as the free energy (or entropy) of the blackhole as evaluated by the Euclidean gravitational action.

$$Z(g_o) \;=\; e^{-I^c_{grav}(g_o)} \;=\; \int [\mathcal{D}A] e^{-I_{YM}(A,g_o)}$$

- Therefore it is essential to understand the relation, in the presence of higher derivative corrections to the Einstein action, between the Euclidean computation of black hole entropy and Wald's prescription in asymptotically AdS space time.

- Plan of the talk
  - A Brief Overview of the Euclidean Method and Wald's Approach (Noetherian method).
  - Correction to the entropy from higher derivative terms.
  - Wald's approach $\equiv$ Euclidean method in asymptotically AdS space.
  - Hydrodynamics on Horizon ?
  - Summary.

Euclidean Black Hole Thermodynamics

- The canonical partition function is defined by a functional integral over metrics with the Euclidean time coordinate τ identified with a fixed period β,

$$Z = \int [\mathcal{D}g]\, e^{-I_E} \qquad where, I_E\ is\ the\ Euclidean\ action.$$

- In the semi-classical limit the dominant contribution to the path integral comes from classical solutions to the EOM.

$$Log Z = -I_E^{cl}, \qquad I_E^{cl}\ evaluated\ on\ solution$$

- Therefore the energy (or mass) of the black hole is given by,

$$E = -\frac{\partial LogZ}{\partial\beta} = \frac{\partial I_E^{cl}}{\partial\beta}$$

- Entropy of the black hole is given by,

$$S = \beta\, E - I_E^{cl}.$$

- For Asymptotically AdS space (Einstein-Hilbert action)

$$E \sim \tilde{R}^4 + f(r_+, b) \qquad\qquad S = \frac{Area}{4G_5}\tilde{f}(r_+, b)$$

- For asymptotically AdS black holes, the prescription needs to be modified.

- The Prescription is following
  - Subtract the contribution of global AdS.
  - BH space time has fixed β. For AdS β' is arbitrary. Adjust the period β' of the globally AdS spacetime such that the geometry at the hypersurface $r = \tilde{R}$ is the same in both cases, i.e,

$$\beta \left(g_{tt}^{BH}\right)^{1/2}\Big|_{r=\tilde{R}} = \beta' \left(g_{tt}^{AdS}\right)^{1/2}\Big|_{r=\tilde{R}} .$$

- prescription ...
  - Partition Function is given by,

$$\boxed{Log(Z) = -\left(I^{cl}_{BH} - I^{cl}_{AdS}\right) \equiv \Delta I}$$

  - Energy

$$E = -\frac{\partial Log(Z)}{\partial \beta} = \frac{\partial \Delta I}{\partial \beta}$$

  - Entropy

$$\boxed{S = \beta E - \Delta I}$$

  - This prescription gives correct answer for AdS Schwarzschild blackhole.

Noetherian Entropy

- Wald approach consists of three steps
  - Step 1. Construction of Noether Charge
  - Step 2. Hamiltonian as Noether Charge
  - Step 3. Entropy as Noether Charge
- Step 1. Construction of Noether Charge
  - For a diffeomorphism invariant Lagrangian **L**, the variation is given by,

$$\delta \mathbf{L} = \mathbf{E}(\psi)\delta\psi + d\mathbf{\Theta}(\delta\psi).$$

  - For diffeomorphism, $\delta\psi = \mathcal{L}_\xi \psi$,

$$\delta \mathbf{L} = \mathcal{L}_\xi \mathbf{L} = d(\xi \cdot \mathbf{L}).$$

- We can define a current

$$\mathbf{J}_\xi = \mathbf{\Theta}(\mathcal{L}_\xi \psi) - \xi \cdot \mathbf{L}.$$

satisfying

$$d\mathbf{J}_\xi = -\mathbf{E}\mathcal{L}_\xi \psi$$

- Now, for any conserved J_ξ there exists a "Noether Charge" $\mathbf{Q}_\xi(\psi)$ constructed out of fields ψ and ξ^a, such that whenever ψ satisfies the equation of motion, we have,

$$\boxed{\mathbf{J}_\xi = d\mathbf{Q}_\xi(\psi)}$$

- Step 2. Hamiltonian as Noether Charge
  - Variation of the Hamiltonian corresponding to a diffeomorphism ξ

$$\delta H_\xi = \int_{\partial\Sigma} (\delta \mathbf{Q}_\xi - \xi \cdot \mathbf{\Theta}(\delta\psi)).$$

  - If there exists some B such that $\mathbf{\Theta}(\delta\psi) = \delta \mathbf{B}(\psi)$

$$H_\xi \sim \int_{\partial\Sigma} \left(Q_\xi - \xi \cdot B\right)$$

  - For $\boxed{\xi = \frac{\partial}{\partial t}}$

$$\mathcal{E} \sim \int_{\partial\Sigma} \left(Q_t - t \cdot B\right)$$

- Step 3. Entropy as Noether Charge
  - Start from the relation,

$$\delta \mathbf{J}_\xi \;=\; \delta \mathbf{\Theta}(\mathcal{L}_\xi \psi) \;-\; \xi \cdot \delta \mathbf{L}(\psi)$$

  - Integrate it over blackhole spacetime and take $\xi = \frac{\partial}{\partial t}$

$$\delta \int_{\mathcal{H}} dS_{ab} \sqrt{-g} Q^{ab} = \delta \mathcal{E}$$

  - It turns out to be consistent to make an identification with the entropy S via,

$$\frac{\kappa}{2\pi} \delta S = \delta \int_{\mathcal{H}} dS_{ab} \sqrt{-g} Q^{ab}, \quad \kappa \;\; \rightarrow \;\; surface \;\; gravity$$

- After making this identification we get the first law of black hole thermodynamics,

$$\frac{\kappa}{2\pi}\delta S = \delta \mathcal{E}.$$

- If the Lagrangian **L** only depends on Riemann tensors and their powers then entropy is given by,

$$\boxed{S = -2\pi \int_{\mathcal{H}} \frac{\partial L}{\partial R_{abcd}} \epsilon_{ab}\epsilon_{cd},}$$

where ϵ_{ab} is the binormal to the surface $\mathcal{H}$.

α' Correction to the Entropy

- Three examples.
  - Gauss-Bonnet term - simplest extension

$$\Delta I = \frac{\alpha'/4}{16\pi G_5} \int d^5x\sqrt{-g}\,(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2)$$

  - $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$

$$\Delta I = \frac{\alpha'/4}{16\pi G_5} \int d^5x\sqrt{-g}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$$

- and the third example is
  - The Type IIB R^4 or equivalently $(Weyl)^4$ term

$$\Delta I = -\frac{\gamma}{16\pi G_{10}} \int d^{10}x \sqrt{-g_{10}} W$$

where

$$W = C^{hmnk} C_{pmnq} C_h{}^{rsp} C^q{}_{rsk} + \frac{1}{2} C^{hkmn} C_{pqmn} C_h{}^{rsp} C^q{}_{rsk}.$$

and

$$\gamma = \frac{1}{8} \zeta(3) (\alpha')^3$$

- Consider the correction terms as a perturbation.
- Find the perturbed solution for these three cases.
- The unperturbed (solution of the Einstein-Hilbert action) solution we will take is

$$ds^2 = \left(1 + \frac{r^2}{b^2} - \frac{\omega M}{r^2}\right) d\tau^2 + \frac{dr^2}{\left(1 + \frac{r^2}{b^2} - \frac{\omega M}{r^2}\right)} + r^2 d\Omega_3^2.$$

- Correction to the metric
  Take the spherically symmetric form of the metric

$$ds^2 = A(r)dt^2 + \frac{1}{B(r)}dr^2 + r^d\Omega_3^2$$

  - For Gauss-Bonnet term

$$A(r) = B(r) = 1 + \frac{r^2}{b^2} - \frac{\omega M}{r^2} + \alpha'\left(\frac{r^2}{2b^4} + \frac{\omega^2 M^2}{2r^6}\right)$$

  - For $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ term

$$A(r) = B(r) = 1 + \frac{r^2}{b^2} - \frac{\omega M}{r^2} + \alpha'\left(\frac{r^2}{6b^4} + \frac{M^2\omega^2}{2r^6}\right)$$

- and
  - For $(Weyl)^4$ term:

$$A(r) = \left(1 + \frac{r^2}{b^2} - \frac{r_+^4 + b^2 r_+^2}{b^2 r^2}\right)\left(1 + \frac{\gamma M^3}{r^{12} r_+^{12}} f_1(r)\right)$$

and

$$B(r) = \left(1 + \frac{r^2}{b^2} - \frac{r_+^4 + b^2 r_+^2}{b^2 r^2}\right)\left(1 + \frac{\gamma M^3}{r^{12} r_+^{12}} f_2(r)\right)$$

$f_{(1|2)}(r) =$
$(45|285) r_+^{12} - 35 b^2 r_+^2 r^2 \frac{r^{10} - r_+^{10}}{r^4 + r^2 b^2 - M b^2} - 75 r_+^4 r^4 \frac{r^8 - r_+^8}{r^4 + r^2 b^2 - M b^2}$
where r_+ is horizon radius, $B(r_+)^{-1} = 0$.

- Correction to the blackhole temperature
  - For Gauss-Bonnet term

$$\beta = \beta_o \left(1 + \frac{\alpha'}{r_+^2}\right).$$

  - For $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ term

$$\beta = \beta_o \left[1 + \frac{\alpha'}{3b^2}\frac{2r_+^4 + 3b^4 + 6b^2 r_+^2}{r_+^2(2r_+^2 + b^2)}\right]$$

  - For $(Weyl)^4$ term -

$$\beta = \beta_o \left[1 - \frac{10\gamma}{b^6}\left(1 + \frac{b^2}{r_+^2}\right)^3 \frac{3r_+^2 - b^2}{2r_+^2 + b^2}\right]$$

Correction to the Entropy

- Entropy for Einstein-Hilbert action is given by,

$$S = \frac{V_3 r_+^3}{4G_5} \equiv S_0$$

- Correction to the entropy for these three cases are given by (using subtraction procedure)
  - For Gauss-Bonnet term
    $$S = S_o \left[1 + \alpha' \frac{3}{r_+^2}\right]$$
  - For $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ term
    $$S = S_o \left[1 + \alpha' \frac{2}{b^2} \left(1 + \frac{3}{2}\frac{b^2}{r_+^2}\right)\right]$$
  - For $(Weyl)^4$ term
    $$S = S_o \left[1 + \frac{60\gamma}{b^6} \left(1 + \frac{b^2}{r_+^2}\right)^3\right]$$

- Entropy calculation using Wald's Formula
  - Contribution to the entropy (leading term) coming from Einstein-Hilbert part is,

$$S_0 = \frac{V_3 r_+^3}{4G_5}$$

  - For $(Weyl)^4$ term correction to the entropy is given by,

$$\Delta S = -\frac{\gamma}{8G_5}\int_{\mathcal{H}} d^3x\sqrt{h}\frac{\partial W}{\partial R_{\mu\nu\rho\sigma}}\epsilon_{\mu\nu}\epsilon_{\rho\sigma}.$$

- Using unperturbed metric one can calculate,

$$\frac{\partial W}{\partial R_{\mu\nu\rho\sigma}}\epsilon_{\mu\nu}\epsilon_{\rho\sigma}\Big|_{r=r_+} = -\frac{120}{b^6}\left(1+\frac{b^2}{r_+^2}\right)^3$$

- So total entropy is,

$$S = S_o\left[1+\frac{60\gamma}{b^6}\left(1+\frac{b^2}{r_+^2}\right)^3\right]$$

Which matches with Euclidean calculation.

- So having seen the computations in both approaches agree nontrivially in a number of examples, we have reason to believe that this is true in general.
- So in the next section we will argue why the Wald's approach is equivalent to Euclidean method.

Wald Approach $\equiv$ Euclidean method

- Mass in asymptotically AdS spacetime
  - Recall the relation

$$\boxed{\delta H_\xi = \delta \int_{\tilde{R}} dS_{ab} \sqrt{-g}\ \ Q^{ab}[\xi]}$$

    (we have dropped the boundary term, but one can go with that term also.)
  - Integrating this equation we get the total energy,

$$H_\xi = \int_{\tilde{R}} dS_{ab} \sqrt{-g}\ \ Q^{ab}[\xi] - \int_{\tilde{R}} dS_{ab} \sqrt{-g}\ \ Q^{ab}_{AdS}[\tilde{\xi}]$$

  - The additive constant has vanishing variation.

- Mass in AdS space continued..
  - This constant has been chosen in such a way so that the Hamiltonian is zero for pure AdS.
  - And the boundary geometry at $r = \tilde{R}$ must be the same for both the BH spacetime and the background AdS, i.e.

$$|\tilde{\xi}|^2 = |\xi|^2 \;\; on \;\; the\ boundary\ hypersurface.$$

  - Since $Q[\xi]$ is linear in ξ, and since the difference in normalization between ξ and $\tilde{\xi}$ is a constant, we can write,

$$\boxed{Q^{ab}_{AdS}[\tilde{\xi}] = \left[\left(\frac{g_{tt}}{g^{AdS}_{tt}}\right)^{\frac{1}{2}}\right]_{r=\tilde{R}} Q^{ab}_{AdS}[\xi].}$$

- The Hamiltonian which is the total energy or mass of the system is then given by

$$M = \int_{\tilde{R}} dS_{ab}\sqrt{-g}\ \ Q^{ab}[t]$$

$$- \left[\left(\frac{g_{tt}}{g_{tt}^{AdS}} \right)^{\frac{1}{2}} \right]_{r=\tilde{R}} \int_{\tilde{R}} dS_{ab}\sqrt{-g}\ \ Q^{ab}_{AdS}[t].$$

- It is clear that for pure AdS spacetime mass is zero.

- ## Euclidean Entropy and Wald Entropy
  - Consider ξ to be a killing vector vanishing on the horizon, so

$$\boxed{\mathbf{J} = -\xi \cdot \mathbf{L}.}$$

  - Integrate both sides of this Eq. over a constant time hypersurface $\mathcal{C}$ of the black hole spacetime, having the interior boundary $\mathcal{H}$ and the outer boundary at $r = \tilde{R}$ and using the above definition of mass we get,

$$\int_{\mathcal{H}} dS_{ab}\sqrt{-g}\ \ Q^{ab}[\xi^t] = M + \int_{\mathcal{C}} dV_t \xi^t L(g_{BH})$$

$$+ \left[\left(\frac{g_{tt}^{BH}}{g_{tt}^{AdS}}\right)^{\frac{1}{2}}\right]_{r=\tilde{R}} \int_{\tilde{R}} dS_{ab}\sqrt{-g}\ \ Q^{ab}_{AdS}[\xi^t].$$

- Now for a constant time hypersurface Σ of global AdS.

$$\int_{\tilde{R}} dS_{ab}\sqrt{-g}\ Q^{ab}_{AdS}[\xi^t] = -\int_{\Sigma} dV_t \xi^t L(g_{AdS}),$$

- Finally we have,

$$S = \beta M + \beta \int_{\mathcal{C}} dV_t \xi^t L(g_{BH})$$

$$-\beta \left(\frac{g_{tt}}{g^{AdS}_{tt}}\right)^{\frac{1}{2}} \int_{\Sigma} dV_t \xi^t L(g_{AdS}).$$

- For a static background, Euclidean action corresponds to

$$I_{BH} = -\beta \int_{\mathcal{C}} dV_t \xi^t L(g_{BH})$$

and also for AdS,

$$I_{AdS} = -\beta \left(\frac{g_{tt}}{g_{tt}^{AdS}} \right)^{\frac{1}{2}} \int_{\Sigma} dV_t \xi^t L(g_{AdS})$$

- And we assign a temperature for global AdS

$$\beta' = \beta \left(\frac{g_{tt}}{g_{tt}^{AdS}} \right)^{\frac{1}{2}},$$

- We can now see exactly the Euclidean prescription,

$$\boxed{S = \beta M - I_{BH} + I_{AdS}.}$$

- Thus starting from the Noetherian expression for entropy, we obtain the relation to the Euclidean prescription with exactly the same subtraction procedure.

Hydrodynamics on Horizon?

- So far we have studied the holographic description of thermodynamics.
- But one can extend thermodynamics to hydrodynamics - low energy fluctuation from thermodynamics.
- Now it is natural to ask the question in the same spirit of thermodynamics whether one can give two kind of holographic description of hydrodynamics of the gauge theory using AdS/CFT prescription?
  - One on the boundary,
  - and the other one on the horizon.

- So far most of the computations deal with the asymptotic boundary. Hydrodynamic characteristic (like shear viscosity) of gauge theory can be obtained by calculating two point correlation functions of gauge theory stress tensors.
- To describe the hydrodynamics on the inner boundary it is better to look at the membrane paradigm of the blackhole. In this view point the blackhole (stretched)horizon acts like a fluid membrane having the hydrodynamics characteristics.
- One can define a membrane stress tensor for the stretched horizon. Though from this membrane stress tensor we get correct the shear viscosity coefficient and also $\frac{\eta}{s}$ ratio comes out to be $\frac{1}{4\pi}$.

- But There are few problems with this picture.
- This description gives a negative bulk viscosity! where the boundary gauge theory being a conformal theory has zero bulk viscosity.
- Also this membrane stress tensor is not sensitive about the asymptotic behavior of the space time which is important for us.
- Our aim is to write down a membrane stress tensor from which one can read of the correct hydrodynamics characteristics of the gauge theory.
- To incorporate the AdS character of the space time we are trying to add some extra term (counter term?) in the action, so that it gives correct viscosity coefficients.

- Also we are interested to find out some general formula in presence of higher derivative correction, which is defined on the horizon, for viscosity coefficients like Wald's formula for entropy.

Summary

- We studied the relation between two holographic descriptions.
- We checked in several cases and found nontrivial agreement between Wald and Euclidean approach for entropy.
- Finally we gave a general argument why these two approaches should always agree in asymptotically AdS space time.
- Trying to find out hydrodynamic description of gauge theory from horizon point of view.