Equivalence of Tensor Product and Partner Hamiltonian Formalism in Supersymmetric Quantum Mechanics

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Abstract. In this article we first write a brief review of supersymmetric quantum mechanics and then we discuss the equivalence of two co-existing formalisms viz. tensor product formalism and partner hamiltonian formalism for 1-D SUSY harmonic oscillator. We also present a Mathematica code with which one can calculate the eigenstates of any 1-D SUSY partner Hamiltonian along with two illustrated examples of 1-D SUSY harmonic oscillator and 1-D SUSY infinite potential box. Then we calculate the SUSY partner wave function for 1-D anharmonic oscillator using this code and plot first few of them. Finally, we give an example how the supersymmetric partner potentials can be calculated starting from a well-behaved ground state wave function [1].

Keywords. Supersymmetry , elementary particles , equivalence of formalisms , quantum mechanics

1. SUPERSYMMERIC QUANTUM MECHANICS

The promising idea of supersymmetry in physics started becoming the point of attraction in the late twenties. The main idea here is to consider a broader picture of the standard model in particle physics by considering bosons and fermions in the same footing. This idea has the potential of solving many problems beyond standard model and in order to that it brought a new kind of symmetry into the picture. This new symmetry allows one to interchange between two seemingly very different kind of particles, bosons and fermions and it brings a new conserved quantity with it namely supercharge. The simplest case of SUSY quantum mechanics is 1D SUSY harmonic oscillator. There exists two parallel formalisms [5][3] for this system in the literature and both of them solve the problem uniquely. In this article we will discuss about how both of these formalisms are deeply related and will point out the equivalence of these two formalisms. At last we will also provide a Mathematica code to calculate and plot the eigenfunction of 1D supersymmetric partner Hamiltonian and present three examples of 1-D harmonic oscillator, 1-D infinite potential well and 1-d inverted harmonic oscillator with an anharmonic term αx^6.

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[1]SUSY, supersymmetry; 1-D, 1 Dimensional; HO, Harmonic Oscillator; AHO, Anharmonic Oscillator; inf. pot., infinite potential

2. A BRIEF REVIEW OF TWO FORMALISMS FOR SUSY HARMONIC OSCILLATOR

Bosonic and fermionic harmonic oscillators are main building block of many physical theories. However, there is a mojor difference between the behaviour of these two particles[2]. By definition, bosons have integeral spin and fermions have half integeral spin. Moreover, according to Pauli exclusion principle no two identical fermions can occupy the same state but there is no such constraint for bosons. Also, we know that under the exchange of two identical fermions the wave function describing the state of these two particles takes up a minus sign but if we exchange two bosons no such minus sign appears in the wave function.
Now, to bring the bosonic and fermionic particles in the same footing 'supersymmetry' plays a crucial role and to incorporate them in a single frame one needs to build a common Hilbert space for both of them. There are mainly two ways of constructing the Hilbert space of this system that yield two different formalisms.

2.1 *Tensor product formalism*

The bosonic harmonic oscillator resides in a Hilbert space,$\mathcal{H}_{\mathcal{B}}$ that is $L^2(\mathcal{R})$ in nature and the Hilbert space of fermionic harmonic oscillator,$\mathcal{H}_{\mathcal{F}}$ is a $\mathcal{C}^2$ space. The ladder operators of bosonic harmonic oscillator are defined by their commutator relations and the ladder operator of fermionic harmonic oscillator are defined by their anti-commutator relation. Now one of the ways to construct the Hilbert space of SUSY harmonic oscillator is by going to a tensor product space of these two systems as

$$\mathcal{H}_{\mathcal{S}} = \mathcal{H}_{\mathcal{B}} \otimes \mathcal{H}_{\mathcal{F}}. \tag{1}$$

The basis of this Hilbert space is defined as

$$|n\rangle_S = |n\rangle_B \otimes |n\rangle_F,$$

where $|n\rangle_B$ and $|n\rangle_F$ are the number state basis of bosonic and fermionic harmonic oscillators respectively. An operator in this Hilbert space is defined as

$$\mathcal{O}_1 \otimes \mathcal{O}_2 : \mathcal{H}_{\mathcal{B}} \otimes \mathcal{H}_{\mathcal{F}} \rightarrow \mathcal{H}_{\mathcal{B}} \otimes \mathcal{H}_{\mathcal{F}}. \tag{2}$$

The Hamiltonian of the bosonic harmonic oscillator can be written as

$$H_B = \hbar\omega_B \left(a^\dagger a + \frac{1}{2}\right), \tag{3}$$

where $a^\dagger$ and a are respectively annihilation and creation operators defined as

$$a^\dagger = \frac{1}{\sqrt{2m\hbar\omega_B}} \left(-ip + m\omega_B x\right) \tag{4}$$

$$a = \frac{1}{\sqrt{2m\hbar\omega_B}} \left(ip + m\omega_B x\right). \tag{5}$$

The commutation relation between them is given by

$$\left[a, a^\dagger\right] = 1, [a, a] = 0, \quad \left[a^\dagger,\ a^\dagger\right] = 0, \tag{6}$$

and $N_B = a^\dagger a$ is the bosonic number operator. This operator acting on the n-th number state gives the number of bosonic particles of that particular state as the eigenvalue.

$$N_B|n\rangle_B = n_B|n\rangle_B, \qquad n_B = 0, 1, 2, .. \tag{7}$$

The energy eigenvalue equation for the above state can be written as

$$H_B|n\rangle_B = E_{n_B}|n\rangle_B = \left(n_B + \frac{1}{2}\right)\hbar\omega|n\rangle_B. \tag{8}$$

The operation of the creation and annihilation operators on the number states is given by

$$a^\dagger|n\rangle_B = \sqrt{n_B + 1}|n+1\rangle_B \tag{9}$$

$$a|n\rangle_B = \sqrt{n_B}|n-1\rangle_B. \tag{10}$$

On the other hand the Hamiltonian of fermionic harmonic oscillator is given by

$$H_F = \hbar\omega_F\left(c^\dagger c - \frac{1}{2}\right), \tag{11}$$

where c and $c^\dagger$ are respectively the fermionic annihilation and creation operator that satisfy the anti-commutation relation as

$$\{c, c^\dagger\} = 1, \quad \{c^\dagger, c^\dagger\} = 0, \quad \{c, c\} = 0. \tag{12}$$

The fermionic number operator is similarly defined as

$$N_F = c^\dagger c, \tag{13}$$

which acts on the n-th fermionic number state as

$$N_F|n\rangle_F = n_F|n\rangle_F, \qquad n_F = 0, 1. \tag{14}$$

The energy eigen value equation for this state would be written as

$$H_F|n\rangle_F = E_{n_F}|n\rangle_F = \left(n_F - \frac{1}{2}\right)\hbar\omega|n\rangle_F. \tag{15}$$

One property of fermionic creation and annihilation operators is that due to their anti-commutation relations, they are nilpotent of order 2 which means

$$\{c^\dagger, c^\dagger\}|n\rangle_F = 0 \tag{16}$$

$$\left(c^\dagger c^\dagger + c^\dagger c^\dagger\right)|n\rangle_F = 0$$

$$c^\dagger c^\dagger|n\rangle_F = -c^\dagger c^\dagger|n\rangle_F$$

$$c^\dagger c^\dagger|n\rangle_F = 0. \tag{17}$$

Similarly for the annihilation operator

$$cc|n\rangle_F = 0, \tag{18}$$

which only leaves two possible state in the fermionic ladder namely $|0\rangle$ and $|1\rangle$ that satisfies

$$c|0\rangle = 0 \ \ \& \ \ c^\dagger|1\rangle = 0. \tag{19}$$

Now we construct the Hilbert space of the joint system as of Eq.1 and keeping in mind the form of the operators in this Hilbert space as of Eq.2 we write the Hamiltonian of the Hilbert space of SUSY harmonic oscillator as

$$H_S = H_B \otimes \mathcal{I}_\mathcal{F} \ + \mathcal{I}_\mathcal{B} \otimes H_B \tag{20}$$

$$= \hbar\omega_B \left(a^\dagger a + \frac{1}{2}\right) \otimes \mathcal{I}_\mathcal{F} \ + \ \mathcal{I}_\mathcal{B} \otimes \left(c^\dagger c - \frac{1}{2}\right) \hbar\omega_F, \tag{21}$$

where $\mathcal{I}_\mathcal{B}$ and $\mathcal{I}_\mathcal{F}$ are the identities of bosonic and fermionic Hilbert spaces respectively. By using the number operator of respective spaces the energy eigenspectrum of the Hilbert space of SUSY harmonic oscillator becomes

$$E = E_B + E_F = \left(n_B + \frac{1}{2}\right) \hbar\omega_B \ + \ \left(n_F - \frac{1}{2}\right) \hbar\omega_F. \tag{22}$$

Likewise, We can define the number operator of this SUSY harmonic oscillator as

$$N_S = N_B \otimes \mathcal{I}_F \ + \ \mathcal{I}_B \otimes N_F. \tag{23}$$

We now come to the main part of the supersymmetry and define an operator Q called Supercharge and its conjugate $Q^\dagger$ as

$$Q = a \otimes c^\dagger \qquad \text{and} \qquad Q^\dagger = a^\dagger \otimes c. \tag{24}$$

These two operators acting on the number state of the Hilbert Space $\mathcal{H}_\mathcal{S}$ yield

$$Q^\dagger|n\rangle_S = \left(a^\dagger \otimes c\right)|n\rangle_B \otimes |n\rangle_F = |n+1\rangle_B \otimes |n-1\rangle_F \tag{25}$$

$$Q|n\rangle_S = \left(a \otimes c^\dagger\right)|n\rangle_B \otimes |n\rangle_F = |n-1\rangle_B \otimes |n+1\rangle_F. \tag{26}$$

These operators change one fermion to one boson and vice versa. So these two operators are called the generators of the supersymmetry. Now in Eq.1 if we take a simplifying assumption that $\omega_B = \omega_F = \omega$, H_S takes the form

$$H_S = \hbar\omega \left(\left(a^\dagger a + \frac{1}{2}\right) \otimes \mathcal{I}_\mathcal{F} \ + \ \mathcal{I}_\mathcal{B} \otimes \left(c^\dagger c - \frac{1}{2}\right)\right)$$

$$H_S = \hbar\omega \left(a^\dagger a \otimes \mathcal{I}_\mathcal{F} \ + \ \mathcal{I}_\mathcal{B} \otimes c^\dagger c\right). \tag{27}$$

Now it can be very easily shown that $Q^\dagger$ and Q commutes with H_S i.e.

$$[H_S, Q] = \left[H_S, Q^\dagger\right] = 0. \tag{28}$$

This implies that the system posses symmetry under the exchange of bosons and fermions and the supercharge is also a conserved quantity. Q and $Q^\dagger$ also obey the following anti-commutator relations

$$\{Q, Q^\dagger\} = H_S, \;\; \{Q, Q\} = 0 \;\text{ and }\; \{Q^\dagger, Q^\dagger\} = 0. \tag{29}$$

2.2 *Super-Potential Formalism*

This is a more general formalism and can handle various 1D SUSY systems unlike the tensor product formalism. Here the trick is also to factorise the Hamiltonian. We begin with 1D time independent Schrodinger's equation of the system of our interest as

$$H_1\psi_1^{(n)} = \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_1(x)\right)\psi_1^{(n)} = E_1^{(n)}\psi_1^{(n)} \tag{30}$$

where $E_1^{(n)}$ and $\psi_1^{(n)}$ are the nth eigenvalue and eigenfunction of H_1. Therefore, we can express the potential in terms of the ground state eigenfunction and eigenvalue as

$$V_1(x) = \frac{\hbar^2}{2m}\frac{1}{\psi_1^{(0)}(x)}\frac{d^2\psi_1^{(0)}}{dx^2} + E_1^{(0)} \tag{31}$$

Now by defining $H_1 - E_1^{(0)}$ as H_B we can write it as

$$H_1 - E_1^{(0)} = H_B \tag{32}$$

$$= -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{\hbar^2}{2m}\frac{1}{\psi_1^{(0)}(x)}\frac{d^2\psi_1^{(0)}}{dx^2} \tag{33}$$

$$= -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_B, \quad \text{where } V_B = \frac{\hbar^2}{2m}\frac{1}{\psi_1^{(0)}(x)}\frac{d^2\psi_1^{(0)}}{dx^2}. \tag{34}$$

At this point we introduce two operators A and $A^\dagger$ in the following manner

$$\begin{aligned} A^\dagger &= -\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x) \\ A &= \frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x), \end{aligned} \tag{35}$$

where $W(x)$ is known as Superpotential. With a little algebra we get

$$A^\dagger A = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \frac{\hbar}{\sqrt{2m}}W'(x) + W^2(x). \tag{36}$$

We can therefore factorise the Hamiltonian H_B as

$$H_B = A^\dagger A, \tag{37}$$

with the identification

$$V_B = W^2(x) - \frac{\hbar}{\sqrt{2m}} W'(x) = V_1 - E_1^{(0)}. \tag{38}$$

We denote the eigenvalues and eigenfunction of H_B as $E_B^{(n)}$ and $\psi_B^{(n)}$ respectively. Note that $\psi_B^{(n)}$ and $\psi_1^{(n)}$ are same and the eigenvalues $E_B^{(n)}$ are different from $E_1^{(n)}$ by a constant shift of $E_1^{(0)}$. Now comes the SUSY part and we define what we call "Partner Hamiltonian" of H_B as

$$H_F = AA^\dagger. \tag{39}$$

Using definition of A and $A^\dagger$ from Eq.35 we can write this equation as

$$H_F = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_F(x) \tag{40}$$

$$\text{where, } V_F(x) = W^2(x) + \frac{\hbar}{\sqrt{2m}} W'(x). \tag{41}$$

We can denote the nth eigenvalue and eigenfunction of H_F as $E_F^{(n)}$ and $\psi_F^{(n)}$. These states posses some beautiful relations which will be very useful later. Note that using Eq.37 and Eq.39 we get

$$H_B\{A^\dagger \psi_F^{(n)}(x)\} = A^\dagger H_F \psi_F^{(n)}(x) = E_F^{(n)}\{A^\dagger \psi_F^{(n)}(x)\} \tag{42}$$

$$H_F\{A\psi_B^{(n)}(x)\} = AH_B \psi_B^{(n)}(x) = E_B^{(n)}\{A\psi_B^{(n)}(x)\} \tag{43}$$

This shows that $A^\dagger \psi_F^{(n)}(x)$ is an eigenstate of H_B and $A\psi_B^{(n)}(x)$ is an eigenstate of H_F. So $A^\dagger$ and A are intertwining operators that link the eigenstates of the two partner Hamiltonians H_B and H_F. With little algebra it can be shown that

$$\psi_F^{(n)} = \left(E_B^{(n+1)}\right)^{-1/2} A\psi_B^{(n+1)} \tag{44}$$

$$\psi_B^{(n+1)} = \left(E_F^{(n)}\right)^{-1/2} A^\dagger \psi_F^{(n)} \tag{45}$$

$$E_F^{(n)} = E_B^{(n+1)} \tag{46}$$

Now in this formalism to show the supersymmetric invariance of the system we go to direct sum space of $\mathcal{H}_\mathcal{B}$ and $\mathcal{H}_\mathcal{F}$ where Hamiltonian H_B belongs to $\mathcal{H}_\mathcal{B}$ Hilbert space and Hamiltonian H_F belongs to $\mathcal{H}_\mathcal{F}$ Hilbert space. So, we define the new Hilbert space and the Hamiltonian as

$$\mathcal{H}_\mathcal{S} = \mathcal{H}_\mathcal{B} \oplus \mathcal{H}_\mathcal{F} \qquad \text{and} \tag{47}$$

$$H_S = \begin{pmatrix} H_B & 0 \\ 0 & H_F \end{pmatrix} \tag{48}$$

and the super charge operator and its conjugate as

$$Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \text{ and } Q^\dagger = \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix}. \tag{49}$$

Now it is easy to show that these operators follow the same commutation and anti-commutation rules as of Eq.28 and Eq.29. We can write the SUSY wave function as

$$\psi_S^{(n)} = \begin{pmatrix} \psi_B^{(n)} \\ \psi_F^{(n)} \end{pmatrix} \tag{50}$$

but note that this is not an eigenstate of H_S due to Eq.46. We shall have to take the state as

$$\psi_S^{(n)} = \begin{pmatrix} \psi_B^{(n)} \\ \psi_F^{(n-1)} \end{pmatrix} \tag{51}$$

to make it an eigenstate of H_S. Now at this point the two formalisms seem to be using different techniques to deal with the problem of SUSY harmonic oscillator. So, in the next section we would like to discuss and illustrate how they are related and how the partner eigenstates are related to the number state of tensor product formalism.

3. EQUIVALENCE OF TENSOR PRODUCT AND PARTNER HAMILTONIAN FORMALISM FOR 1-D SUSY HARMONIC OSCILLATOR

To understand the equivalence of these two formalisms we have to first understand what $\psi_F^{(n)}$ and $\psi_B^{(n)}$ means physically. A system is called fermionic when the total spin of the system is half integral and we note that in 1-D harmonic oscillator the number of bosons can range from 0 to any large value but the number of fermion can be either 0 or 1. From Eq.32 we see that for 1-D harmonic oscillator $\psi_B^{(n)}$ has energy eigenvalue

$$E_B^{(n)} = E_1^{(n)} - E_1^{(0)} = n\hbar\omega. \tag{52}$$

This directly implies that there are total n bosons and 0 fermions in this state which is equivalent to $|n\rangle \otimes |0\rangle$ state of tensor product formalism or the nth bosonic excitation state of SUSY Hamiltonian of tensor product space. The fermionic partner Hamiltonian state $\psi_F^{(n)}$ refers a state where we have n bosons and 1 fermion making the total spin half integral and hence it is equivalent to the $|n\rangle \otimes |1\rangle$ state of tensor product formalism. By this comparison we can see clear physical meaning of Eq.46. From Eq.52 and Eq.46 we get $E_F^{(n)} = (n+1)\hbar\omega$. Using this idea of equivalence we can find it to be trivial that $\psi_F^{(n)}$ is a state containing n+1 particles and therefore, its energy is same as the energy of $|n\rangle \otimes |1\rangle$ state. From Eq.44 and Eq.45 we can now understand the operation of A and $A^\dagger$ on the partner eigenstates more clearly that they respectively create a fermion by destroying a boson and vice versa. This property was not that clear from equation 35. The normalisation constant of Eq.44 and Eq.45 can be calculated very easily. To derive Eq.44, Let us assume that

$$\psi_F^{(n)} = c_1 A \psi_B^{(n+1)} \tag{53}$$

where c_1 is the normalisation constant then by taking inner products we get

$$1 = c_1^2 \langle \psi_B^{(n+1)} | A^\dagger A | \psi_B^{(n+1)} \rangle \tag{54}$$

$$= c_1^2 \langle \psi_B^{(n+1)} | H_B | \psi_B^{(n+1)} \rangle \tag{55}$$

$$= c_1^2 E_B^{(n+1)} \tag{56}$$

$$\text{or,} c_1 = \left(E_B^{(n+1)} \right)^{-1/2} \tag{57}$$

we can do this similarly for Eq.45. Now we present a schematic diagram for visualising the connections between these formalisms and their corresponding states and energies.

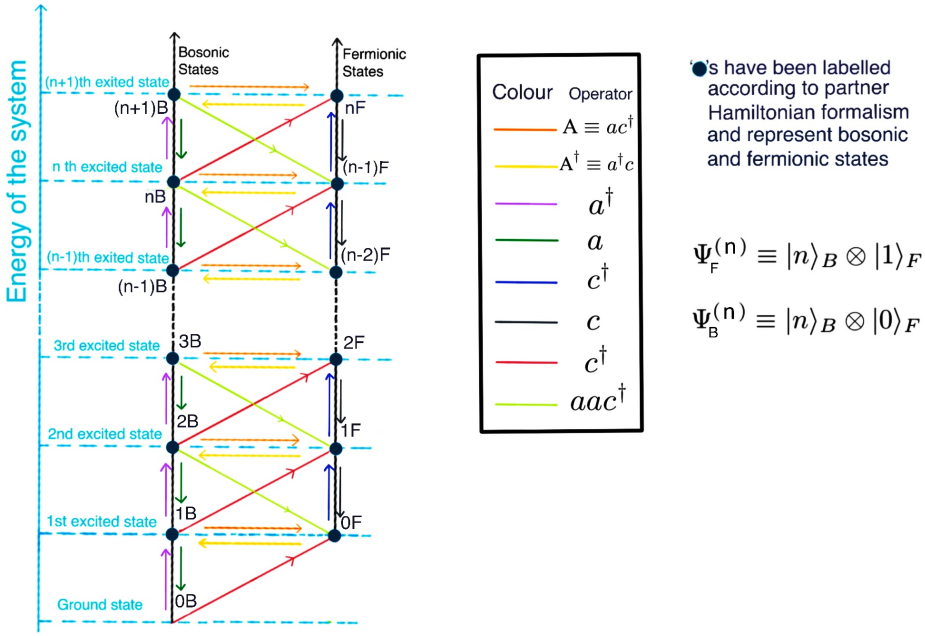

Figure 1. Schematic diagram of equivalence of tensor product and partner Hamiltonian formalism

In the partner Hamiltonian formalism the ground state is a bit special as there is no fermionic partner eigenstate for this and the ground state of SUSY harmonic oscillator is bosonic. In the diagram we have a pair of black vertical lines. The nodes on the left one of them represents the bosonic states whereas the nodes on the right one denotes the fermionic states and these states are

equidistant, marked by their corresponding level of excitation.
Here we give a table showing the equivalence of states of the two formalisms and their corresponding positions marked in the diagram. All the arrows represent the necessary operators for going from one state to another one.

Point in the diagram	Cor. TP state	Eqv. PH wave fn.	E of the state	n_B of the state	n_F of the state
0B	$\|0\rangle_B \otimes \|0\rangle_F$	$\psi_B^{(0)}$	0	0	0
0F	$\|0\rangle_B \otimes \|1\rangle_F$	$\psi_F^{(0)}$	$\hbar\omega$	0	1
1B	$\|1\rangle_B \otimes \|0\rangle_F$	$\psi_B^{(1)}$	$\hbar\omega$	1	0
1F	$\|1\rangle_B \otimes \|1\rangle_F$	$\psi_F^{(1)}$	$2\hbar\omega$	1	1
2B	$\|2\rangle_B \otimes \|0\rangle_F$	$\psi_B^{(2)}$	$2\hbar\omega$	2	0
2F	$\|2\rangle_B \otimes \|1\rangle_F$	$\psi_F^{(2)}$	$3\hbar\omega$	2	1
3B	$\|3\rangle_B \otimes \|0\rangle_F$	$\psi_B^{(3)}$	$3\hbar\omega$	3	0
(n-2) F	$\|n-2\rangle_B \otimes \|1\rangle_F$	$\psi_F^{(n-2)}$	$(n-1)\hbar\omega$	n-2	1
(n-1)B	$\|n-1\rangle_B \otimes \|0\rangle_F$	$\psi_B^{(n-1)}$	$(n-1)\hbar\omega$	n-1	0
(n-1) F	$\|n-1\rangle_B \otimes \|1\rangle_F$	$\psi_F^{(n-1)}$	$n\hbar\omega$	n-1	1
n B	$\|n\rangle_B \otimes \|0\rangle_F$	$\psi_B^{(n)}$	$n\hbar\omega$	n	0
n F	$\|n\rangle_B \otimes \|1\rangle_F$	$\psi_F^{(n)}$	$(n+1)\hbar\omega$	n	1
(n+1) B	$\|n+1\rangle_B \otimes \|0\rangle_F$	$\psi_B^{(n+1)}$	$(n+1)\hbar\omega$	n+1	0

Cor., Corresponding; TP, Tensor Product; Eqv., Equivalent; PH, Partner Hamiltonian;
fn, function; E, Energy; n_B, number of bosons; n_F, number of fermions;

Table 1. Table of equivalence

So we note that the index 'n' in partner Hamiltonian formalism irrespective of ψ_B or ψ_F represents the number of bosons in that state. Now as A destroys a boson and creates a fermion, for the bosonic ground state we can write

$$A\psi_B^{(0)}(x) = 0. \tag{58}$$

$$\text{This implies,} \quad W(x) = -\frac{\hbar}{\sqrt{2m}} \frac{1}{\psi_B^{(0)}} \frac{d\psi_B^{(0)}}{dx} \tag{59}$$

$$= -\frac{\hbar}{\sqrt{2m}} \frac{d\ln\left(\psi_B^{(0)}\right)}{dx}. \tag{60}$$

So, this equation completes the calculation of SUSY wave function for 1D cases as using equations 35, 44, 46, 60 we can calculate the partner wave functions of any 1-D SUSY quantum mechanical system as we show in the next section.

4. CALCULATION OF SUSY PARTNER WAVE FUNCTIONS FOR 1-D SUSY HARMONIC OSCILLATOR

In this section we calculate the eigenfunction of the fermionic Hamiltonian H_F. Now for 1-D harmonic oscillator we know that

$$H_1\psi_1^{(n)} = \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2\right)\psi_1^{(n)} = E_1^{(1)}\psi_1^{(n)}. \tag{61}$$

The eigenfunction and and the energy eigenvalues are given by

$$\psi_1^{(n)} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}\exp\left(-\frac{m\omega}{2\hbar}x^2\right)H_n\left[\left(\frac{m\omega}{\hbar}\right)^{1/2}x\right] \tag{62}$$

where, $H_n(x)$ are Hermite polynomials and

$$E_1^{(n)} = \left(n + \frac{1}{2}\right)\hbar\omega. \tag{63}$$

So, as we have discussed earlier that $\psi_1^{(n)} = \psi_B^{(n)}$ and $E_B^{(n)} = n\hbar\omega$. The ground state wave function of $\psi_B^{(n)}$ and its derivative is given by

$$\psi_B^{(0)} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right) \text{ and } \frac{d}{dx}\psi_B^{(0)} = -x\frac{m\omega}{\hbar}\psi_B^{(0)}. \tag{64}$$

From Eq.60 we get

$$W(x) = \frac{\hbar}{\sqrt{2m}}\frac{m\omega}{\hbar}x = \sqrt{\frac{m}{2}}\omega x. \tag{65}$$

So, now from Eq.41 we can write

$$V_F(x) = \frac{m\omega^2}{2}x^2 + \frac{1}{2}\hbar\omega, \tag{66}$$

and from Eq.46 we get

$$E_F^{(n)} = (n+1)\hbar\omega. \tag{67}$$

Now, from Eq.44 and Eq.35 we can write the form of eigenstate of H_F as

$$\psi_F^{(n)} = \frac{1}{\sqrt{E_B^{(n+1)}}}A\psi_B^{(n+1)} \tag{68}$$

$$= \frac{1}{\sqrt{(n+1)\hbar\omega}}\left(\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x)\right)\left(\mathcal{N}\exp\left(-\frac{m\omega x^2}{2\hbar}\right)H_n\left[\left(\frac{m\omega}{\hbar}\right)^{1/2}x\right]\right), \tag{69}$$

where $\mathcal{N}$ is the corresponding normalisation constant.

Now after doing the simplification we get

$$\psi_F^{(n)} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} \exp\left(-\frac{m\omega}{2\hbar}x^2\right) H_n\left[\left(\frac{m\omega}{\hbar}\right)^{1/2} x\right]. \tag{70}$$

We see that the form of eigenfunctions are same for H_B and H_F in the case of 1-D SUSY harmonic oscillator which is a well known example of shape invariant potential [1] in supersymmetric quantum mechanics.

5. MATHEMATICA CODE

In this section we present a Mathematica code using which one can calculate the eigenfunctions of H_B and H_F for any 1-D SUSY quantum mechanical system. The plots of the eigenfunctions have been made in two different ways. Firstly, we have used the superpotential formalism in order to obtain $\psi_F^{(n)}$ from $\psi_B^{(n)}$ as shown in previous section. Secondly, we have used the analytical solution of the partner eigenfunction to plot it.The first method is a more general numerical way of obtaining the supersysmmetric partner eigenfunction even for the cases when it is hard to solve the schrodinger equation of the system analytically.
We can observe that the plot of the wavefunctions made in two different ways matches perfectly.

SUSY 1 - D Harmonic Oscillator

```
In[•]:= Clear["Global`*"];
```

■ Value of the Parameters : -

```
In[•]:= ħ = 1; m = 1/2; ω = 1;
```

■ Potential : -

```
In[•]:= V₁[x_] := 1/2 m ω^2 x^2;
```

■ Hamiltonian : -

```
In[•]:= H₁ = -ħ^2/(2 m) * u''[x] + V₁[x] * u[x];
```

■ Obtaining Eigenvalues and Eigenfunctions (Ground State to 5th Excited State) of 1 D Harmonic Oscillator : -

```
In[•]:= {Eigenvalue, Eigenfunction} = NDEigensystem[H₁, u[x], {x, -100, 100}, 6,
          Method →
           {"SpatialDiscretization" ->
             {"FiniteElement", {"MeshOptions" -> {"MaxCellMeasure" -> 0.01}}}}];
```

Out[•]=

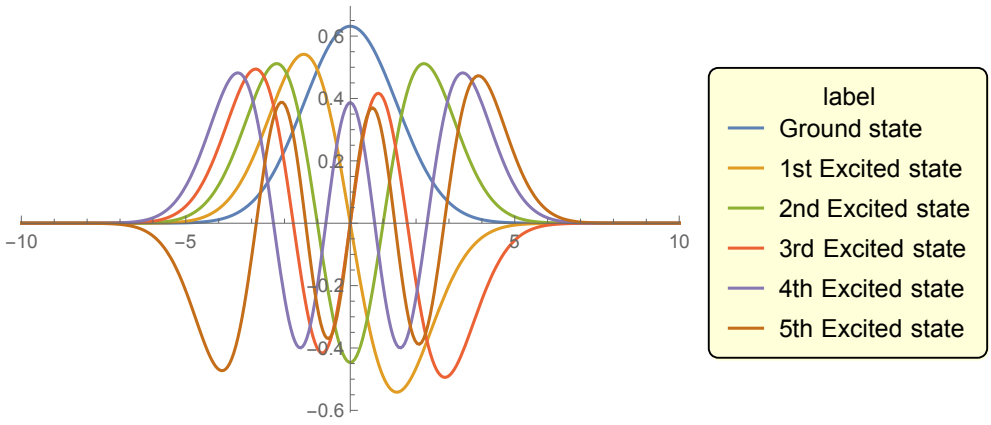

Figure 2. Numerical plots of the first 6 eigenstates of bosonic HO hamiltonian

■ Defining : -

```
In[•]:= E₁[n_] := Eigenvalue[[n]];
```

NOTE : Array indexing starts from 1 in Mathematica. So n = 1 is the Ground State in this case.

■ Energy Eigenvalue and Eigenfunction of Bosonic Hamiltonian H_B : –

```
In[•]:= E_B[n_] := E_1[n] - E_1[1];
        ψ_B[n_] := Eigenfunction[[n]];
```

■ Superpotential : -

```
In[•]:= W = -ψ_B[1]^(-1) * D[ψ_B[1], x];
```

■ Fermionic Eigenfunction or Eigenfunction of the Partner Hamiltonian (H_F) : –

$$\blacksquare\ \psi_F^{(n)} = \left(E_B^{(n+1)}\right)^{-\frac{1}{2}} \left(\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x)\right) \psi_B^{(n+1)}$$

```
In[•]:= ψ_F[n_] := (E_B[n + 1])^(-1/2) * (ħ/√(2m)) D[ψ_B[n + 1], x] + W * ψ_B[n + 1]
```

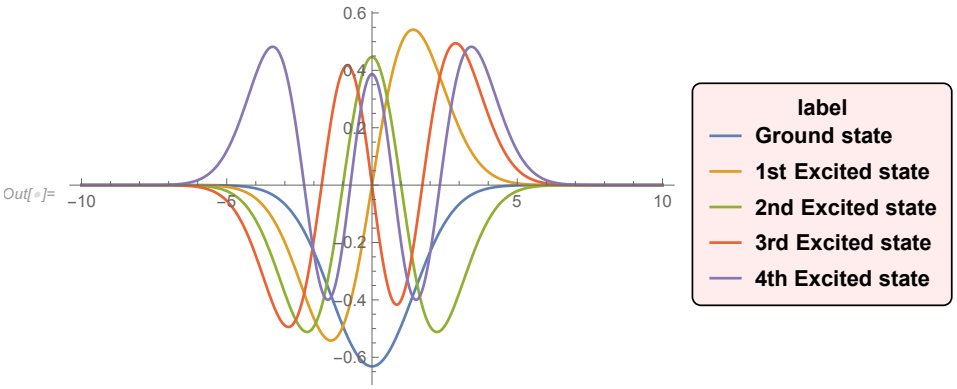

Figure 3. Numerical plots of the first 5 eigenstates of fermionic HO hamiltonian

The Analytic Solution of the Partner Hamiltonian Eigenfunction : -

■ Parameters : -

```
In[•]:= m = 1/2; ω = 1; ħ = 1;
```

Now by using the form of Eq. 70 we define,

```
In[•]:= ϕF[n_, x_] := (m*ω/π)^(1/4) 1/√(2^n n!) Exp[-(m*ω)/(2 ħ) x^2] HermiteH[n, (m*ω/ħ)^(1/2) x]
```

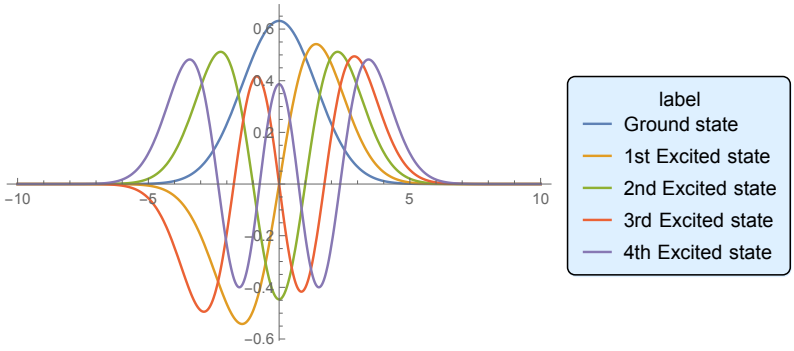

Figure 4. Plots of the first 5 analytical solutions of HO eigenstates

NB : The plots of ψ_F and ϕ_F are same apart from a random sign flip. This sign is actually inherited from the method that Mathematica uses for solving the differential equations. However it doesn't matter because if ψ_F is an eigenfunction of H_F then $(-\psi_F)$ is also an eigenfunction of H_F with the same eigenvalue.

Changing the value of the potential $V_1(x)$ we can plot the partner eigenfunction for any 1D SUSY Quantum System. Below we are showing the same thing for an 1D Infinite Potential Box. For the analytic solution we have considered the form of the partner eigenfunction as given in [4].

SUSY 1 D InfinitePotential Box

■ Values of the Parameters : -

```
In[•]:= ħ = 1; m = 1/2;
```

■ Potential : -

```
In[•]:= V1[x_] := 0;
```

■ Hamiltonian : -

```
In[•]:= H1 = -ħ^2/(2 m) * u''[x] + V1[x] * u[x];
```

■ Boundary Condition : -

```
In[•]:= B = DirichletCondition[u[x] == 0, True];
```

■ Obtaining Eigenvalues and Eigenfunctions(Ground State to 5th Excited State) of 1D Infinite Potential Box :-

```
In[•]:= {Eigenvalue, Eigenfunction} = NDEigensystem[{H1, B}, u[x], {x, 0, 1}, 6,
          Method → {"Eigensystem" → {"Arnoldi",
              "MaxIterations" → 10000},
            "PDEDiscretization" → {"FiniteElement", "MeshOptions" → {"MaxCellMeasure" → 0.001}}}];
```

Out[•]=

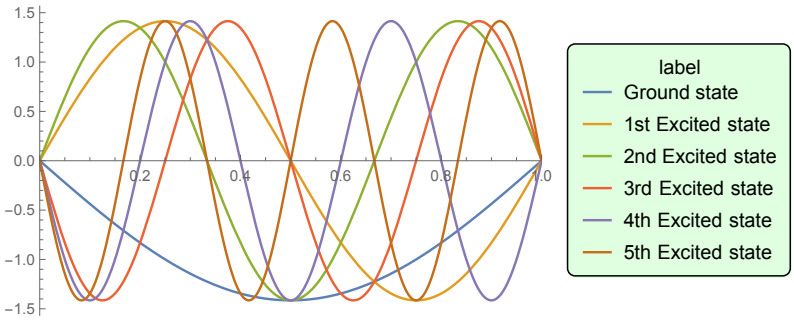

Figure 5. Numerical plots of the first 6 eigenstates of bosonic HO hamiltonian

Defining:-

```
In[•]:= E₁[n_] := Eigenvalue[[n]];
```

■ Energy Eigenvalue and Eigenfunction of Bosonic Hamiltonian H_B : –

```
In[•]:= E_B[n_] := E₁[n] - E₁[1];
        ψ_B[n_] := Eigenfunction[[n]];
```

■ Superpotential : -

```
In[•]:= W = -ψ_B[1]^(-1) * D[ψ_B[1], x];
```

■ Fermionic Eigenfunction or Eigenfunction of the Partner Hamiltonian (H_F) : –

$$\blacksquare\ \psi_F^{(n)} = \left(E_B^{(n+1)}\right)^{-\frac{1}{2}} \left(\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x)\right) \psi_B^{(n+1)}$$

```
In[•]:= ψ_F[n_] := (E_B[n + 1])^(-1/2) * (ħ/√(2 m)) D[ψ_B[n + 1], x] + W * ψ_B[n + 1]
```

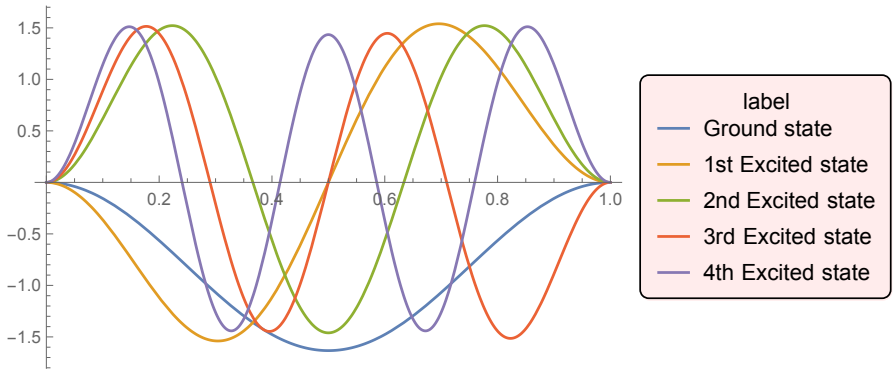

Figure 6. Numerical plots of the first 5 eigenstates of fermionic 1-D inf. pot. box hamiltonian

The Analytic Solution of the Partner Hamiltonian Eigenfunction : -

■ Parameter : -

```
In[•]:= L = 1;
```

```
In[•]:= ϕF[n_, x_] := √(2 / (L ((n + 2) ^2 - 1))) ((n + 2) * Cos[(n + 2) π/L x] - Cot [π/L x] * Sin [(n + 2) π/L x])
```

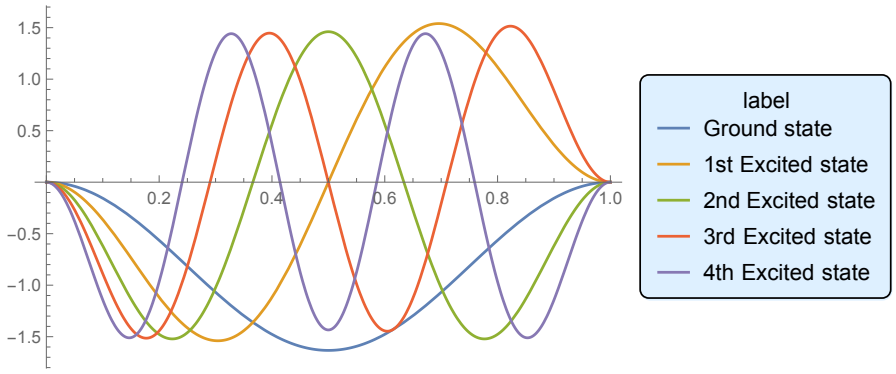

Figure 7. Plots of the first 5 analytical solutions of 1-D inf. pot. box hamiltonian eigenstates

Now we present the results for a case which can not be studied analytically without perturbation theory, the inverted harmonic oscillator. Here we have used the following form of potential;

$$V_1(x) = -\frac{1}{2}m\omega^2 x^2 + \alpha x^6 \tag{71}$$

Using this code we can very easily calculate its partner wave function numerically. We have here plotted only first few of them.

SUSY 1 - D Anharmonic Oscillator

■ Value of the Parameters : -

```
In[•]:= ħ = 1; m = 1/2; ω = 1; α = 1;
```

■ Potential : -

```
In[•]:= V1[x_] := -1/2 m ω^2 x^2 + α x^6;
```

■ Hamiltonian : -

```
In[•]:= H1 = -ħ^2/(2 m) * u''[x] + V1[x] * u[x];
```

■ Obtaining Eigenvalues and Eigenfunctions (Ground State to 3rd Excited State) of 1D Harmonic Oscillator : -

```
In[•]:= {Eigenvalue, Eigenfunction} =
          NDEigensystem[H1, u[x], {x, -100, 100}, 4,
           Method →
            {"SpatialDiscretization" ->
              {"FiniteElement", {"MeshOptions" -> {"MaxCellMeasure" -> 0.01}}}}];
```

Out[•]=

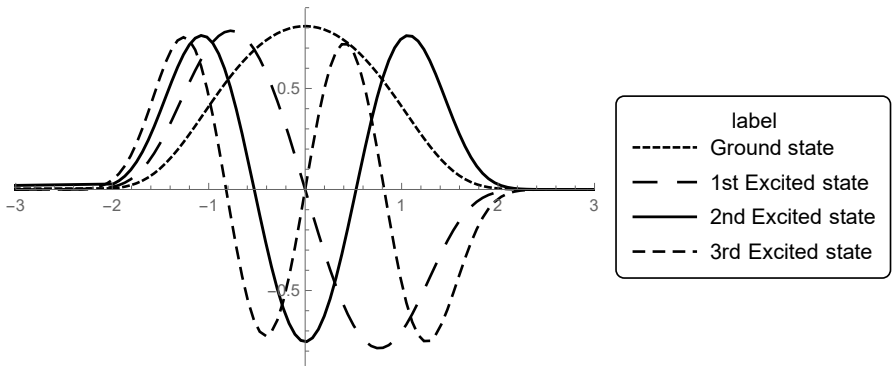

Figure 8. Numerical plots of the first 4 eigenstates of bosonic AHO hamiltonian

■ Defining : -

```
In[•]:= E1[n_] := Eigenvalue[[n]];
```

NOTE : Array indexing starts from 1 in Mathematica. So n = 1 is the Ground State in this case.

■ Energy Eigenvalue and Eigenfunction of Bosonic Hamiltonian H_B : –

```
In[•]:= E_B[n_] := E_1[n] - E_1[1];
        ψ_B[n_] := Eigenfunction[[n]];
```

■ Superpotential : -

```
In[•]:= W = -ψ_B[1]^(-1) * D[ψ_B[1], x];
```

■ Fermionic Eigenfunction or Eigenfunction of the Partner Hamiltonian (H_F) : –

■ $\psi_F^{(n)} = \left(E_B^{(n+1)}\right)^{-\frac{1}{2}} \left(\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x)\right) \psi_B^{(n+1)}$

```
In[•]:= ψ_F[n_] := (E_B[n + 1])^(-1/2) * (ħ/√(2 m)) D[ψ_B[n + 1], x] + W * ψ_B[n + 1]
```

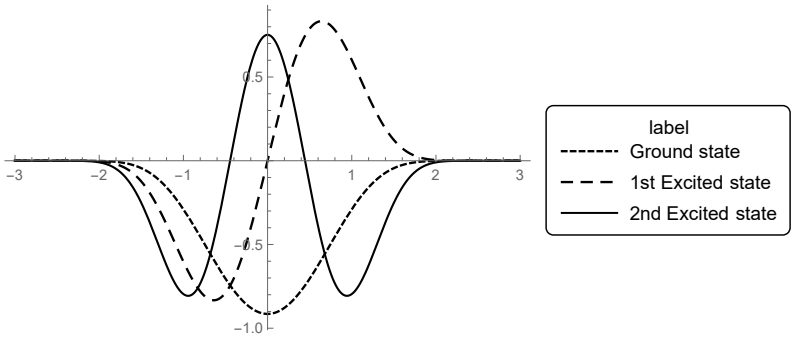

Figure 9. Numerical plots of the first 3 eigenstates of fermionic AHO hamiltonian

6. CALCULATION OF POTENTIAL FROM A SUITABLE GROUND STATE

Now we deal with a different problem where we first take a suitable ground state wave function and then try to find out what form of potential would be needed to generate this ground state wave function and moreover, we can also solve the Schrodinger's equation for that potential to find out higher excited eigenstates and their corresponding eigenvalues. So for that purpose we take the ground eigenstate of the form,

$$\psi = Ae^{-Bx^4} \tag{72}$$

and normalize it. By normalising we get,

$$\psi = \left[\frac{(2B)^{\frac{1}{4}}}{2\Gamma\left(\frac{5}{4}\right)}\right]^{\frac{1}{2}} e^{-Bx^4} \tag{73}$$

where $\Gamma(n)$ is the well known Gamma function. Now we can calculate W using Eq.60 as,

$$W = -\frac{1}{\sqrt{2}}\frac{d}{dx}(\ln\psi) = \frac{4}{\sqrt{2}}Bx^3 \tag{74}$$

where we have considered $\hbar = m = 1$. Then using Eq.38 and Eq.41, we can obtain V_B and V_F as,

$$V_B = W^2 - \frac{1}{\sqrt{2}}W' = 8B^2x^6 - 6Bx^2 \tag{75}$$

$$V_F = W^2 + \frac{1}{\sqrt{2}}W' = 8B^2x^6 + 6Bx^2 \tag{76}$$

Now to get a standard form and to proceed with the numerical calculations, we choose B=$\frac{1}{12}$ and we write V_B and V_F as,

$$V_B = \frac{1}{18}x^6 - \frac{1}{2}x^2 \tag{77}$$

$$V_F = \frac{1}{18}x^6 + \frac{1}{2}x^2 \tag{78}$$

This potential can be substituted in the Mathematica code presented in the previous section to get the energy eigenvalues and the higher energy eigenstates. Analytical and numerical solutions of Schrodinger's equation for similar kind of potential have been studied by several authors [6–9]. They have considered the general problem with the potential $ax^6 - bx^2$ and have calculated the exact analytical or numerical solutions for specific values of a and b.

7. CONCLUSION

In this paper we have discussed the equivalence of partner Hamiltonian and tensor product formalism which is an important idea to understand the framework of SUSY 1-D quantum mechanical systems. The Mathematica code can be used to get the partner eigenstates for very general 1-D potentials also for which we can get perturbative analytic solutions only as we have shown in the example of inverted harmonic oscillator with anharmonic term .

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