Demonstration of a Quantum Harmonic Oscillator by involving Higher Qubit States and further associating it to a Bosonic System

Rajdeep Tah[1,*], Bikash K. Behera[2] and Prasanta K. Panigrahi[2†]
[1]3rd Semester (2nd Year), Integrated MSc., School of Physical Sciences
[1]National Institute of Science Education and Research, Bhubaneswar, HBNI, P.O. Jatni, Khurda-752050, Odisha, India.
[2]Department of Physical Sciences
[2]Indian Institute of Science Education and Research Kolkata, Mohanpur- 741246, West Bengal, India

Abstract. Here, we simulate a discretized quantum oscillator on a digital quantum computer provided by the IBM quantum experience platform. The simulation is carried out in two spatial dimensions and directions are provided for its extension to n-spatial dimensions. Further, we outline the necessary formulations for relevant operators and using them, we perform simulation of a particle in a discretized quantum harmonic oscillator potential using higher qubit system especially a five-qubit system. Finally we attempt to link the QHO (Quantum Harmonic Oscillator) to a Bosonic system and study it. We associated the concept of Pauli Matrix equivalent to Bosonic Particles and used it to calculate the Unitary Operators which helped us to theoretically visualize each Quantum states and further simulate our system.

Keywords. Quantum Harmonic Oscillator, Pauli Matrix equivalents, Bosonic System, Quantum States, Rabi Hamiltonian, Creation and Annihilation Operators, Unitary operator.

1. INTRODUCTION

Harmonic oscillator is one of the most fundamental problems in the field of Physics and it is involved in all aspects of Physics. The reason is still unknown to us but it is very natural for us to understand that whenever a system is disturbed from its minimum energy state then in the course of attaining minimum energy state again, the system will tend to oscillate. This is how a harmonic oscillator functions in a classical sense. Hence, it is worth to search for such a system in the quantum world, too. Thus, Quantum Harmonic Oscillator is nothing but a quantum mechanical analog of the classical harmonic oscillator.

A Quantum Harmonic Oscillator is different from a Classical Harmonic Oscillator mainly on the basis of three grounds: First, the ground energy state for a quantum harmonic oscillator is non-zero because there exists fluctuations as a result of Heisenberg Uncertainty Principle: Second, a particle in a quantum harmonic oscillator potential can be found outside the region -A $\leq x \leq$ +A with a non-zero probability: Thirdly, the probability density distributions for a quantum oscillator in the

*rajdeep.tah@niser.ac.in
†pprasanta@iiserkol.ac.in

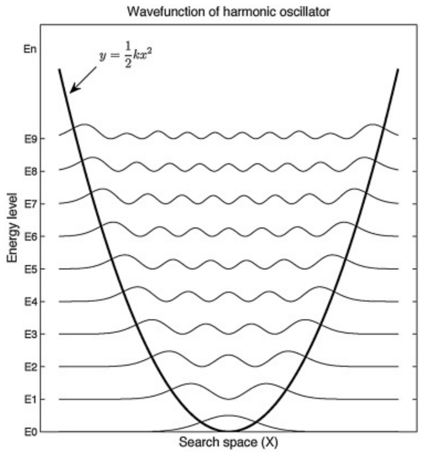

Figure 1. A Generalized Representation of Quantum Harmonic Oscillator

ground low-energy state is largest at the middle of the well. It is commonly used as a model to study the vibrations of the atomic particles and molecules under the effect of classical spring like potential which is a commonly accepted model for the molecular bonding. QHO (quantum harmonic oscillator) is one of the exactly solvable models in the field of quantum mechanics having solutions in the form of Hermite polynomials and it can be generalized to N-dimensions. Its application is not only restricted to the study of simple di-atomic molecule, but it's in fact expanded to the different domains of Physics, e.g. in the study of complex modes of vibration in larger molecule, the theory of heat capacity, QHO as a thermodynamic heat engine, etc.

2. HARMONIC OSCILLATOR IN BRIEF

The most common and familiar version of the Hamiltonian of the Quantum Harmonic Oscillator in general can be written as:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = \frac{\hat{p}^2}{2m} + \frac{1}{2}k\hat{x}^2 \tag{1}$$

Where $\hat{H}$ is the Hamiltonian of the System, m is the mass of the particle, k is the bond stiffness (which is analogous to spring constant in classical mechanics), $\hat{x}$ is the position operator and $\hat{p} = -i\hbar\frac{\partial}{\partial x}$ is the momentum operator (where $\hbar$ is the reduced Plank's constant).
The analytical solution of the Schrodinger wave equation is given by[1]:

$$\Psi = \sum_{n_x=0}^{\infty}\sum_{n_y=0}^{\infty}\frac{1}{2^n\,n!}\left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-\frac{\zeta^2}{2}}e^{-\frac{\beta^2}{2}}H_{n_x}(\zeta)H_{n_y}(\beta)U(t) \tag{2}$$

Where;

$$\zeta = \sqrt{\frac{m\omega}{\hbar}}x \quad \text{and} \quad \beta = \sqrt{\frac{m\omega}{\hbar}}y$$

Here H_n is the nth order Hermite polynomial. U(t) is the Unitary Operator of the system showing its time evolution and is given by:

$$U(t) = exp\left(\frac{-itE_n}{\hbar}\right) = e^{\frac{-itE_n}{\hbar}} \tag{3}$$

Where E_n are the allowed energy eigenvalues of the particle and are given by:

$$E_n = (n_x + \frac{1}{2})\hbar\omega + (n_y + \frac{1}{2})\hbar\omega = (n_x + n_y + 1)\hbar\omega \tag{4}$$

And the states corresponding to the various energy eigenvalues are orthogonal to each other and satisfy:

$$\int_{-\infty}^{+\infty} \psi_j \psi_x dx_i = 0 : \forall\, x_i \tag{5}$$

A much simpler approach to the harmonic oscillator problem lies in the use of ladder operator method where we make use of ladder operators i.e. the creation and annihilation operators $(\hat{b}^\dagger, \hat{b})$, to find the solution of the problem.

Here $\hat{b}^\dagger$ denotes the 'Creation' operator and $\hat{b}$ denotes the 'Annihilation' operator in Bosonic System. We can also the Hamiltonian in terms of the creation and annihilation operators $(\hat{b}^\dagger, \hat{b})$[2]:

$$\hat{H} = \hbar\omega(\hat{b}\hat{b}^\dagger - \frac{1}{2}) = \hbar\omega(\hat{b}\hat{b}^\dagger + \frac{1}{2})$$

Now the Hamiltonian for "a discrete quantum harmonic oscillator" is given by:

$$\hat{H} = \frac{(\hat{p}^d)^2}{2} + \frac{(\hat{x}^d)^2 + (\hat{y}^d)^2}{2} \tag{6}$$

Where $\hat{p}^d$ is the discrete momentum operator and $\hat{x}^d$ and $\hat{y}^d$ are the discrete position operators in in x and y spatial dimension respectively. Also $\hat{p}^d$ can be expressed as:

$$\hat{p}^d = (F^d)^{-1} \cdot \hat{x}^d \cdot (F^d) \tag{7}$$

Where F^d is the standard discrete Quantum Fourier Transform matrix[3].

3. DISCRETIZATION OF THE SPACE

Generally, Discretization is the process of transforming the continuous functions, models and so on into their discrete counterparts. In fact, it is a necessity because to perform any kind of calculation on machines like quantum computers, one needs to work with specific number of elements so that one can obtain outputs which have valid meanings over realistic time scales. However, the Hamiltonian in Eq.(1) allows a continuous eigenspectrum of position. That is why, we have to discretize the space. The Hamiltonian for "a discrete quantum harmonic oscillator" is given by:

$$\hat{H} = \frac{(\hat{p}^d)^2}{2} + \frac{(\hat{x}^d)^2 + (\hat{y}^d)^2}{2} \tag{8}$$

Where $\hat{p}^d$ is the discrete momentum operator and $\hat{x}^d$ and $\hat{y}^d$ are the discrete position operators in x and y spatial dimension respectively. Now let us consider N number of finite elements in a two dimensional space where $x, y \in [-L, +L]$ such that a mesh of N^2 number of elements can be created with each mesh point corresponding to a particular eigenvalue of x and y (A list of 5-qubit states and their corresponding mesh points are given in the following table). Assuming the Harmonic Oscillator potential to be centred at (0,0), we can write the position operator in the form of $N \times N$ matrix with all the position eigenvalues lying along its diagonal as:

$$[\hat{x}^d] = \sqrt{\frac{2\pi}{N}} \begin{bmatrix} -N/2 & 0 & 0 & . & 0 \\ 0 & (-N/2)+1 & 0 & . & 0 \\ . & . & . & . & . \\ 0 & 0 & . & . & . \\ 0 & 0 & 0 & . & (N/2)-1 \end{bmatrix} \tag{9}$$

5-qubit state	Mesh point
$\|00000\rangle$	x_i = -16
$\|00001\rangle$	x_i = -15
$\|00010\rangle$	x_i = -14
.	. .
.	. .
$\|11111\rangle$	x_i = 15

TABLE I. 5-Qubit state vs Mesh Point

The momentum operator ($\hat{p}^d$) can also be calculated in the same way. But to ease our calculation in finding the momentum eigenvalue for each position eigenfunction, we try to make the quantum Fourier transform of the wave function. This helps us to transform the wave function to momentum space where the momentum operators apply multiplicatively and the momentum eigenvalues will be the same as the position eigenvalues for respective discretized space points. An inverse discrete quantum Fourier transform can then be performed to bring back the function into Cartesian-space. The discrete momentum operator can then be applied as a $N \times N$ matrix given by:

$$\hat{p}^d = (\hat{F}^d)^{-1} \cdot \hat{x}^d \cdot (\hat{F}^d) \tag{10}$$

Where $\hat{F}^d$ is the standard discrete Quantum Fourier Transform matrix[3]. Each element of $\hat{F}^d$ can be expressed as:

$$[\hat{F}^d]_{j,k} = \frac{exp(2i\pi jk/N)}{\sqrt{N}} \tag{11}$$

Where $j, k \in [-\frac{N}{2}, ..., \frac{N}{2} - 1]$ and j= no. of rows in the matrix and k= no. of columns in the matrix.
This process is applicable for larger systems and we can have both position and momentum operator. We can also compute the unitary operator for studying the evolution of our system over time.

4. UNITARY TRANSFORMATION

For the sake of reducing mathematical complexity, let us assume $\hbar$, ω and m is unity (i.e. all are having value 1). So, we can write the Schrodinger equation as:

$$i\frac{\partial \Psi}{\partial t} = \hat{H}\Psi \tag{12}$$

Which further implies:

$$\Psi(t) = \Psi(0)exp(-i\hat{H}t) \tag{13}$$

From the above, it is vivid that the Unitary Operator to be computed is $U(t) = exp(-i\hat{H}^d t)$ where; $\hat{H}^d$ is the Discretized Hamiltonian Operator mentioned in Eq.(8). So, the Unitary Operator is given by:

$$U(t) = exp\bigg(-it\bigg(\frac{(\hat{p}^d)^2}{2} + \frac{(\hat{x}^d)^2 + (\hat{y}^d)^2}{2}\bigg)\bigg)$$

Or if we consider the X-dimension only, then we get the Unitary Operator as:

$$U_{\hat{x}}(t) = exp\bigg(\frac{-it}{2}((F^d)^{-1} \cdot \hat{(x^d)}^2 \cdot (F^d) + (\hat{x}^d)^2)\bigg) \tag{14}$$

Due to the discretization of space; the position operator $[\hat{x}^d]$, being a diagonal matrix, can be expanded by using the concept of Matrix exponential as Ref.[4]:

$$exp\bigg(-\frac{it}{2}[A]\bigg) = \mathbb{I} + \sum_{m=1}^{\infty}\bigg(-\frac{it}{2}\bigg)^m \frac{[A]^m}{m!} \tag{15}$$

Here A is the corresponding Operator Matrix and by using the matrix in this form, a time evolution can be performed for any arbitrary state. Such an evolution can be carried out using n-qubits in a quantum circuit.

5. QUANTUM SIMULATION

In our simulation we make use of 5 qubits for our simulation, so we consider 2^5 dimensional Hilbert space (since for a n-qubit system $N = 2^n$). Each dimension in the Hilbert space corresponds to an eigenfunction of a particular position eigenvalue. For two spatial dimensions; $x, y \in [-16, 16]$ and our space is discretized into 32 $\times$ 32 = 1024 individual mesh points. In order to construct quantum gates, we need to compute the unitary operator matrix. From Equation (4), the position operator in its matrix form can be written as:

$$[\hat{x}^d] = \sqrt{\frac{\pi}{16}} \begin{bmatrix} -16 & 0 & 0 & 0 & . & . & . & 0 \\ 0 & -15 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & -14 & 0 & . & . & . & 0 \\ 0 & 0 & 0 & -13 & . & . & . & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & 14 & 0 \\ 0 & 0 & 0 & 0 & . & . & 0 & 15 \end{bmatrix} \tag{16}$$

Previously we have seen that $[\hat{x}^d]$ does not hold the sole importance. Because of the structure of the Hamiltonian, $[\hat{x}^d]^2$ plays a huge role for computing unitary operators for kinetic and potential energy portion individually. We can clearly see that $[\hat{x}^d]^2$ is a diagonal matrix with each diagonal element equal to corresponding position eigenvalue squared and is given by:

$$[\hat{x}^d]' = [\hat{x}^d]^2 = \frac{\pi}{16} \begin{bmatrix} 256 & 0 & 0 & 0 & . & . & . & 0 \\ 0 & 225 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 196 & 0 & . & . & . & 0 \\ 0 & 0 & 0 & 169 & . & . & . & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & 196 & 0 \\ 0 & 0 & 0 & 0 & . & . & 0 & 225 \end{bmatrix} \tag{17}$$

From Eq.(15); the Unitary Operator for the potential energy portion, can be expressed as:

$$U_{\hat{x}}(t) = \mathbb{I} + \left(-\frac{it}{2}\right)^1 \frac{[m]}{1!} + \left(-\frac{it}{2}\right)^2 \frac{[m]^4}{2!} + \left(-\frac{it}{2}\right)^3 \frac{[m]^6}{3!} + ... \tag{18}$$

Where; m = $\hat{x}^d$. We can observe an exact Taylor expansion of the exponential function formed by the addition of the corresponding diagonal elements of the matrix $U_{\hat{x}}$ (after expanding the above expression). We also notice that it is symmetrical about the element at position [16, 16] which is unity.

$$U_{\hat{x}}(t) = \begin{bmatrix} e^{-6.29it} & 0 & 0\;0\;.\;. & . & 0 \\ 0 & e^{-5.52it} & 0\;0\;.\;. & . & 0 \\ 0 & 0 & 0\;.\;.\;. & . & 0 \\ . & . & .\;.\;.\;. & . & . \\ 0 & 0 & 0\;0\;.\;. & e^{-4.81it} & 0 \\ 0 & 0 & 0\;0\;.\;. & 0 & e^{-5.52it} \end{bmatrix} \tag{19}$$

$U_{\hat{x}}(t)[1,1]$	1 - 6.29it - 19.76t^2 + ...	exp(-6.29it)
$U_{\hat{x}}(t)[2,2]$	1 - 5.52it - 15.26t^2 + ...	exp(-5.52it)
$U_{\hat{x}}(t)[3,3]$	1 - 4.81it - 11.58t^2 + ...	exp(-4.81it)
.	....	.
.	....	.
.	....	.
.	....	.
$U_{\hat{x}}(t)[31,31]$	1 - 4.81it - 11.58t^2 + ...	exp(-4.81it)
$U_{\hat{x}}(t)[32,32]$	1 - 5.52it - 15.26t^2 + ...	exp(-5.52it)

TABLE II. Taylor Expansion of Exponential function in $U_{\hat{x}}$.

Now we factor out the element at position [1,1] from the above equation (which kind of acts as a global phase) and we get the ultimate form for the unitary operator as:

$$U_{\hat{x}}(t) = e^{-6.29it} \begin{bmatrix} 1 & 0 & 0\;0\;.\;. & . & 0 \\ 0 & e^{0.77it} & 0\;0\;.\;. & . & 0 \\ 0 & 0 & 0\;.\;.\;. & . & 0 \\ . & . & .\;.\;.\;. & . & . \\ 0 & 0 & 0\;0\;.\;. & e^{1.48it} & 0 \\ 0 & 0 & 0\;0\;.\;. & 0 & e^{0.77it} \end{bmatrix} \tag{20}$$

From the above matrix; we can see that the first diagonal element is independent of any phase term and hence gives an upper edge in the formation of quantum circuit. The rest of the diagonal terms hold the phase values to be provided to the next 31 basis states in the sequence. Therefore, the complete Unitary Operator for our case can be given as:

$$U(t) = exp\left[\left(-\frac{it}{2}\right)(\hat{p}_x^2 + \hat{p}_y^2 + \hat{x}^2 + \hat{y}^2)\right] = U_{\hat{p}_x} U_{\hat{p}_y} U_{\hat{x}} U_{\hat{y}} \tag{21}$$

The orthogonality of the spatial dimensions x and y makes it possible that they obey the same mathematical procedure independently.

GENERALIZATION

The Momentum and Position Operators are orthogonal so they commute with each other which further allows us to generalize our methodology for computing the unitary operator to any arbitrary spatial dimensions. Here we will generalize the calculation of unitary operator for any arbitrary state using any arbitrary number of qubits. The following matrices and tables are relevant for the same:

$$[\hat{x}^d] = \sqrt{\frac{2\pi}{N}} \begin{bmatrix} -N/2 & 0 & 0 & . & . & 0 \\ 0 & (-N/2)+1 & 0 & . & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & . \\ 0 & 0 & 0 & . & 0 & (N/2)-1 \end{bmatrix} \tag{22}$$

$$[\hat{x}^d]^2 = [\hat{x}^d]' = \left(\frac{2\pi}{N}\right) \begin{bmatrix} (-N/2)^2 & 0 & 0 & . & 0 \\ 0 & ((-N/2)+1)^2 & 0 & . & 0 \\ . & . & . & . & . \\ 0 & 0 & . & . & . \\ 0 & 0 & 0 & . & ((N/2)-1)^2 \end{bmatrix} \tag{23}$$

And the Unitary Operator will be:

$$U(t) = exp\left[-\frac{i[\hat{x}^d]^2 t}{2}\right]$$

Which can be written/ expanded using Taylor expansion as:

$$U(t) = exp\left[-\frac{it[\hat{x}^d]^2}{2}\right] = \mathbb{I} + \sum_{m=1}^{\infty}\left(-\frac{it}{2}\right)^m \frac{[[\hat{x}^d]^2]^m}{m!}$$

$$\implies U_{\hat{x}}(t) = \mathbb{I} + \left(-\frac{it}{2}\right)^1 \frac{[\hat{x}^d]^2}{1!} + \left(-\frac{it}{2}\right)^2 \frac{[\hat{x}^d]^4}{2!} + \left(-\frac{it}{2}\right)^3 \frac{[\hat{x}^d]^6}{3!} + \ldots.. \tag{24}$$

To avoid Mathematical complexity and to give the jest of the algorithm, we will drop the term $\left(\frac{2\pi}{N}\right)$ and write the Unitary operator in its matrix form:

$$U_{\hat{x}}(t) = \begin{bmatrix} e^{(-\frac{N}{2})^2 it} & 0 & 0 & . & . & . & 0 \\ 0 & e^{((-\frac{N}{2})+1)^2 it} & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & e^{((-\frac{N}{2})-2)^2 it} & 0 \\ 0 & 0 & 0 & 0 & . & 0 & e^{((-\frac{N}{2})-1)^2 it} \end{bmatrix} \tag{25}$$

6. CIRCUIT IMPLEMENTATION

Using Eq.(21), we can naturally figure out that by making a circuit for $U_x(t)$ and $U_p(t)$ in series, we can complete the quantum circuit for the complete Unitary operator. Initially we will implement

$U_p(t)$ because $U_x(t)$ simply adds the extra phase factor in the circuit. And also to implement $U_p(t)$ we need the Quantum Fourier Transform. So, we first propose a generalized circuit for Quantum Fourier Transform[3] in Fig.(2) which is given in the next page.
Before that we must consider that the basis states enumerate all the possible states of the qubits given as:

$$|x\rangle = |x_1 x_2 \ldots x_n\rangle = |x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_n\rangle \tag{26}$$

Where, with tensor product notation '$\otimes$', $|x_j\rangle$ indicates that qubit j is in state x_j, with x_j either 0 or 1.

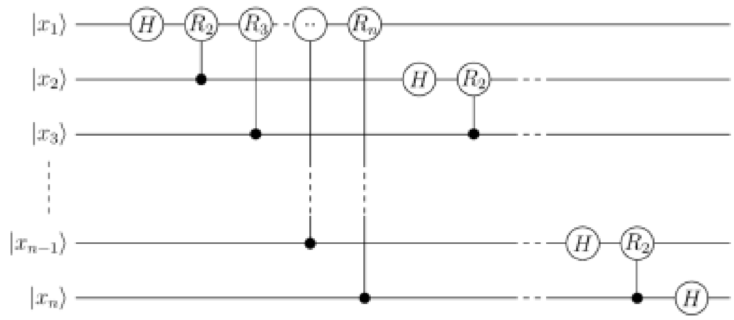

Figure 2. A generalized circuit for the implementation of The Quantum Fourier Transform of n-qubit system.

In the above circuit, the quantum gates used are Hadamard Gate (H) and Controlled Phase Gate (R_m). An efficient quantum circuit for Quantum Fourier Transform of a 5-qubit system is given in Fig.(3) in the next page. The phase of the individual controlled U1 gate is given by:

$$\phi = \frac{2\pi}{2^n}$$

Here, n = number of qubits used in the system. The controlled phase gate (cU1) can thus be written in the form of a matrix:

$$cU1_n = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{2\pi i}{2^n}} \end{bmatrix} \tag{27}$$

With $e^{\frac{2\pi i}{2^n}} = \omega'_n = \omega_{(2^n)}$ which further indicates that $\omega_{(2^n)}$ or ω'_n is the primitive 2^n-th root of unity (or 1).
So, we express H and $cU1_n$ in terms of:

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad cU1_n = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{2\pi i}{2^n}} \end{bmatrix} \tag{28}$$

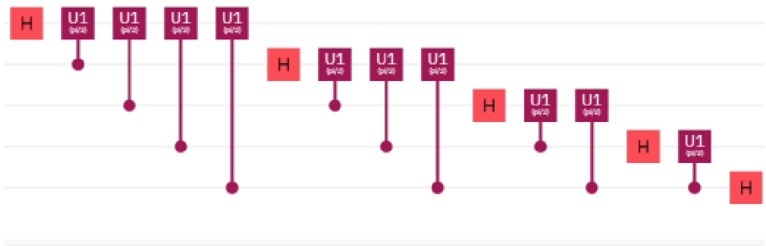

Figure 3. A circuit for implementing the Quantum Fourier Transform using 5-qubit system.

Similarly, in order to implement the inverted Quantum Fourier Transform (QFT), we need to generate a quantum circuit which is the mirror image of Quantum Fourier Transform with conjugate phases at each controlled phase gate. An efficient quantum circuit for the same is given in Fig.(4).

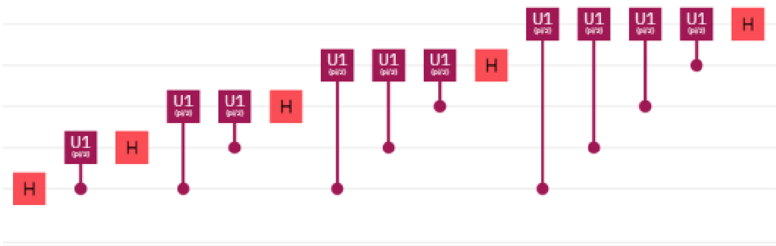

Figure 4. A circuit depicting the Inverted QFT-Modified Mirror image of the circuit for direct Quantum Fourier Transform.

Now the operator $[x^d]'$ which comes in between the QFT and inverted QFT, can be implemented by using a series of Toffolli gate (also known as CCNOT gate or $D(\pi/2)$ gate) and Controlled Phase gate. To make the circuit applicable for all possible states of a 5-qubit system, we need to make circuit in such a way that each of the state kind of gets the correct phase and provides an accurate result. Here, we kind of make the quantum circuit (which is also called a filter) in the specific fashion that the crucial phase information has been imparted. Each set of filters form the state for the next set of filter and a series of such 31 filters can be used recursively to implement all the 31 phase rotations to achieve $U_x(t)$ (or to reduce the number of gates, one can exploit the symmetry about the element positioned at [16,16] by using controlled X gate). A filter for the first state 00001 is illustrated in Fig.(5).

Now by using the same technique, we can prepare the basic quantum circuit for a 3-qubit system. First, we will write the unitary matrix for a 3-qubit system and then we will expand each diagonal element of the unitary operator to make an exact Taylor expansion of the exponential function.

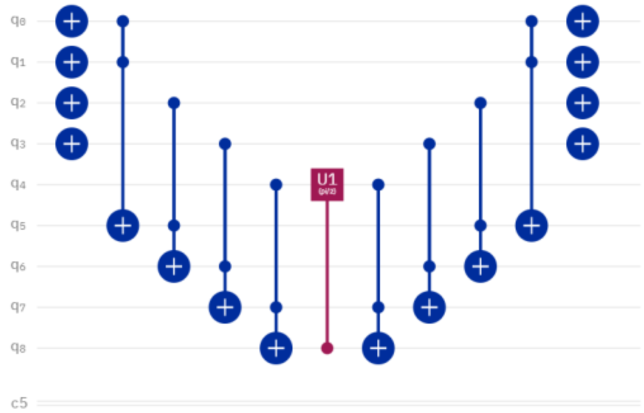

Figure 5. The Inverted QFT-Modified Mirror image of Circuit For Direct Quantum Fourier Transform.

The following matrix is the unitary operator for the 3-qubit system:

$$
U_x^3(t) = \begin{bmatrix} e^{-1.57it} & 0 & 0 & 0\,.\,. & . & 0 \\ 0 & e^{-0.88it} & 0 & 0\,.\,. & . & 0 \\ 0 & 0 & e^{-0.39it} & .\,.\,. & . & 0 \\ . & . & . & .\,.\,. & . & . \\ 0 & 0 & 0 & 0\,.\,. & e^{-0.39it} & 0 \\ 0 & 0 & 0 & 0\,.\,. & 0 & e^{-0.88it} \end{bmatrix}
$$

$$
\implies U_x^3(t) = e^{-1.57it} \begin{bmatrix} 1 & 0 & 0 & 0\,.\,. & . & 0 \\ 0 & e^{0.69it} & 0 & 0\,.\,. & . & 0 \\ 0 & 0 & e^{1.18it} & .\,.\,. & . & 0 \\ . & . & . & .\,.\,. & . & . \\ 0 & 0 & 0 & 0\,.\,. & e^{1.18it} & 0 \\ 0 & 0 & 0 & 0\,.\,. & 0 & e^{0.69it} \end{bmatrix} \tag{29}
$$

Now, clearly by using the quantum Fourier transform and its inverse with our filters in between, we can design and implement the circuit for 3-qubit system. And, U3 gates can be used in the very beginning of the circuit to initialize the circuit to some arbitrary state.
The quantum circuit for the 3-qubit system is shown in Fig.(6).

Figure 6. Quantum circuit for implementing $U_p(t)$ for a 3-qubit system.

7. IMPLEMENTATION ON A BOSONIC SYSTEM

The Hamiltonian of the full system is given by[5]:

$$\hat{H} = \hat{H}_{field} + \hat{H}_{atom} + \hat{H}_{int}$$

where $\hat{H}_{field}$ is the free Hamiltonian, $\hat{H}_{atom}$ is the atomic excitation Hamiltonian and $\hat{H}_{int}$ is the interaction Hamiltonian.

7.1 *MODEL*

We can derive the Pauli Matrix Equivalents for Bosonic System by using the following three equations[6]:

$$(\sigma_3)_{jl} = \frac{\langle s, j|S_k|s, l\rangle}{s\hbar} = \frac{j}{s}\delta_{ij} \tag{30}$$

$$(\sigma_1)_{j\,l} = \frac{[s\,(s+1) - j\,(j-1)]^{1/2}}{2\,s}\,\delta_{j\ l+1} + \frac{[s\,(s+1) - j\,(j+1)]^{1/2}}{2\,s}\,\delta_{j\ l-1} \tag{31}$$

$$(\sigma_2)_{j\,l} = \frac{[s\,(s+1) - j\,(j-1)]^{1/2}}{2\,\mathrm{i}\,s}\,\delta_{j\ l+1} - \frac{[s\,(s+1) - j\,(j+1)]^{1/2}}{2\,\mathrm{i}\,s}\,\delta_{j\ l-1} \tag{32}$$

$\therefore$ By using the above three equations, we have:

$$\sigma_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{33}$$

$$\sigma_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix} \tag{34}$$

$$\sigma_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{35}$$

Where, $\sigma_1, \sigma_2, \sigma_3$ are the Pauli Matrix equivalents for Bosonic particles.

We have modeled our system using Rabi Hamiltonian. However, in our case we will be using somewhat modified version of Rabi Hamiltonian[7]:

$$H_s = \sum_{k=1}^{2} \omega_k b_k^\dagger b_k + \frac{\omega_0}{2}\sigma_3 + \sum_{k=1}^{2} g_k(e^{i\theta_k} b_k + e^{-i\theta_k} b_k^\dagger)\sigma_1 \tag{36}$$

Where ω_0 is the frequency of the main oscillator, ω_k is the frequency of the k-th environment oscillator; $b_k^\dagger$ and b_k are the creation and annihilation operators of the main system and the k-th environmental oscillator respectively. Whereas g_k's are the coupling constant for the interaction between the k-th environment oscillator and the main quantum oscillator. We set k=1 from now to prevent us from complicating the process.
For simplicity, we will consider the simplest case of our model and substitute k=1 in our original Hamiltonian [in Eq.(36)] to obtain the special case of our Hamiltonian which will be our working Hamiltonian from now:

$$H = \omega_1 b_1^\dagger b_1 + \frac{\omega_0}{2}\sigma_3 + g_1(e^{i\theta_1} b_1 + e^{-i\theta_1} b_1^\dagger)\sigma_1$$

For simplicity we will drop the sub-script 1 from our Hamiltonian and obtain:

$$H = \omega b^\dagger b + \frac{\omega_0}{2}\sigma_3 + g(e^{i\theta} b + e^{-i\theta} b^\dagger)\sigma_1 \tag{37}$$

7.2 *RELEVANT TRANSFORMATION AND GENERALIZATION*

Now, as our system involves Bosonic particles, so the following commutation relations uphold:

$$[b_i, b_j^\dagger] \equiv b_i b_j^\dagger - b_j^\dagger b_i = \delta_{ij} \tag{38}$$

$$[b_i^\dagger, b_j^\dagger] = [b_i, b_j] = 0 \tag{39}$$

Here δ_{ij} is known as 'Kronecker delta'.
The operators used in the Hamiltonian can be transformed according to Holstein-Primakoff transformations (i.e. it maps spin operators for a system of spin-S moments on a lattice to creation and annihilation operators) as[8]:

$$\hat{S}_j^+ = \sqrt{(2S - \hat{n}_j)}\hat{b}_j \tag{40}$$

$$\hat{S}_j^- = \hat{b}_j^\dagger \sqrt{(2S - \hat{n}_j)} \tag{41}$$

Where $\hat{b}_j^\dagger$ ($\hat{b}_j$) is the creation (annihilation) operator at site j that satisfies the commutation relations mentioned above and $\hat{n}_j = \hat{b}_j^\dagger \hat{b}_j$ is the "Number Operator". Hence we can generalize the above equations as:

$$S^+ = \sqrt{(2S - b^\dagger b)}b \tag{42}$$

$$S^{-} = b^{\dagger}\sqrt{(2S - b^{\dagger}b)} \tag{43}$$

Where;

$$S_{+} \equiv S_x + iS_y \qquad \text{and} \qquad S_{-} \equiv S_x - iS_y$$

Where; S_x (= σ_1), S_y (= σ_2), S_z (= σ_3) are the Pauli matrices for Bosonic system (as mentioned in the previous section).

Now by using the above transformations; we can write our creation and annihilation operators in terms of Matrices as:

$$b^{\dagger} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad \text{and} \qquad b = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{44}$$

Now the Hamiltonian for our coupled Quantum Harmonic Oscillator in Eq.(37) can be decomposed as:

$$H = \omega b^{\dagger}b \otimes \mathbb{I} + \frac{\omega_0}{2}\mathbb{I} \otimes \sigma_3 + g(e^{i\theta}b + e^{-i\theta}b^{\dagger}) \otimes \sigma_1$$

Or the above equation can be written as:

$$H = \omega b^{\dagger}b \otimes \mathbb{I} + \frac{\omega_0}{2}\mathbb{I} \otimes S_3 + g(e^{i\theta}b + e^{-i\theta}b^{\dagger}) \otimes S_1 \tag{45}$$

Now, we will evaluate each term to simplify the expression of the Hamiltonian in the form of matrix. Here,

$$\omega b^{\dagger}b \otimes \mathbb{I} = \omega \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \omega b^{\dagger}b \otimes \mathbb{I} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \omega & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega \end{bmatrix} \tag{46}$$

Similarly,

$$\frac{\omega_0}{2}\mathbb{I}\otimes S_z = \frac{\omega_0}{2}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\otimes\begin{bmatrix}1&0&0\\0&0&0\\0&0&-1\end{bmatrix}$$

$$\Rightarrow \frac{\omega_0}{2}\mathbb{I}\otimes S_z = \begin{bmatrix}\frac{\omega_0}{2}&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0\\0&0&-\frac{\omega_0}{2}&0&0&0&0&0&0\\0&0&0&\frac{\omega_0}{2}&0&0&0&0&0\\0&0&0&0&0&0&0&0&0\\0&0&0&0&0&-\frac{\omega_0}{2}&0&0&0\\0&0&0&0&0&0&\frac{\omega_0}{2}&0&0\\0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&-\frac{\omega_0}{2}\end{bmatrix} \tag{47}$$

Finally,

$$g(e^{i\theta}b+e^{-i\theta}b^{\dagger})\otimes S_x = \frac{g}{\sqrt{2}}\begin{bmatrix}0&e^{i\theta}&0\\e^{-i\theta}&0&e^{i\theta}\\0&e^{-i\theta}&0\end{bmatrix}\otimes\begin{bmatrix}0&1&0\\1&0&1\\0&1&0\end{bmatrix}$$

$$\Rightarrow g(e^{i\theta}b+e^{-i\theta}b^{\dagger})\otimes S_x = \frac{g}{\sqrt{2}}\begin{bmatrix}0&0&0&0&e^{i\theta}&0&0&0&0\\0&0&0&e^{i\theta}&0&e^{i\theta}&0&0&0\\0&0&0&0&e^{i\theta}&0&0&0&0\\0&e^{-i\theta}&0&0&0&0&0&e^{i\theta}&0\\e^{-i\theta}&0&e^{-i\theta}&0&0&0&e^{i\theta}&0&e^{i\theta}\\0&e^{-i\theta}&0&0&0&0&0&e^{i\theta}&0\\0&0&0&0&e^{-i\theta}&0&0&0&0\\0&0&0&e^{-i\theta}&0&e^{-i\theta}&0&0&0\\0&0&0&0&e^{-i\theta}&0&0&0&0\end{bmatrix} \tag{48}$$

Substituting the above values in Eq.(45), we get the value of H (a 9×9 matrix) as:

$$\Rightarrow H = \begin{bmatrix} \frac{\omega_0}{2} & 0 & 0 & 0 & \frac{ge^{i\theta}}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{ge^{i\theta}}{\sqrt{2}} & 0 & \frac{ge^{i\theta}}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & -\frac{\omega_0}{2} & 0 & \frac{ge^{i\theta}}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & \frac{ge^{-i\theta}}{\sqrt{2}} & 0 & (\omega + \frac{\omega_0}{2}) & 0 & 0 & 0 & \frac{ge^{i\theta}}{\sqrt{2}} & 0 \\ \frac{ge^{-i\theta}}{\sqrt{2}} & 0 & \frac{ge^{-i\theta}}{\sqrt{2}} & 0 & \omega & 0 & \frac{ge^{i\theta}}{\sqrt{2}} & 0 & \frac{ge^{i\theta}}{\sqrt{2}} \\ 0 & \frac{ge^{-i\theta}}{\sqrt{2}} & 0 & 0 & 0 & (\omega - \frac{\omega_0}{2}) & 0 & \frac{ge^{i\theta}}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & \frac{ge^{-i\theta}}{\sqrt{2}} & 0 & (\omega + \frac{\omega_0}{2}) & 0 & 0 \\ 0 & 0 & 0 & \frac{ge^{-i\theta}}{\sqrt{2}} & 0 & \frac{ge^{-i\theta}}{\sqrt{2}} & 0 & \omega & 0 \\ 0 & 0 & 0 & 0 & \frac{ge^{-i\theta}}{\sqrt{2}} & 0 & 0 & 0 & (\omega - \frac{\omega_0}{2}) \end{bmatrix}$$

8. DERIVATION OF UNITARY OPERATORS

Clearly, we know that for a system with Hamiltonian H, the unitary operator is given by:

$$U = e^{-iHt} \tag{49}$$

Where H is the Hamiltonian of the system derived in the previous section.
But to find the unitary operator compatible, we need to change the form of our Hamiltonian and write it as a sum of two matrices whose corresponding unitary operators are relatively easier to compute:

$$H = X + Y$$

Where,

$$X = \begin{bmatrix} \frac{\omega_0}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\omega_0}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (\omega + \frac{\omega_0}{2}) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (\omega - \frac{\omega_0}{2}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (\omega + \frac{\omega_0}{2}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (\omega - \frac{\omega_0}{2}) \end{bmatrix}$$

$$Y = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{ge^{i\theta}}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{ge^{i\theta}}{\sqrt{2}} & 0 & \frac{ge^{i\theta}}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{ge^{i\theta}}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & \frac{ge^{-i\theta}}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & \frac{ge^{i\theta}}{\sqrt{2}} & 0 \\ \frac{ge^{-i\theta}}{\sqrt{2}} & 0 & \frac{ge^{-i\theta}}{\sqrt{2}} & 0 & 0 & 0 & \frac{ge^{i\theta}}{\sqrt{2}} & 0 & \frac{ge^{i\theta}}{\sqrt{2}} \\ 0 & \frac{ge^{-i\theta}}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & \frac{ge^{i\theta}}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & \frac{ge^{-i\theta}}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{ge^{-i\theta}}{\sqrt{2}} & 0 & \frac{ge^{-i\theta}}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{ge^{-i\theta}}{\sqrt{2}} & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus we have,

$$U = e^{-iXt}.e^{-iYt}$$
$$\implies U = U_x(t).U_y(t)$$

Where $U_x(t) = e^{-iXt}$ and $U_y(t) = e^{-iYt}$. First we will compute $U_y(t)$, then $U_x(t)$. We can see that $U_y(t)$ can be expanded using Taylor series of expansion of the exponential function as:

$$U_y(t) = exp(-itY) = \mathbb{I} + \sum_{m=1}^{\infty} (-it)^m \frac{Y^m}{m!}$$

$$\implies U_y(t) = \mathbb{I} + (-it)^1 \frac{Y}{1!} + (-it)^2 \frac{Y^2}{2!} + (-it)^3 \frac{Y^3}{3!} + (-it)^4 \frac{Y^4}{4!} + (-it)^5 \frac{Y^5}{5!} + ...$$

Now, for simplicity, let us denote $\frac{g}{\sqrt{2}} = g^{'}$. So, we have:

$$\implies U_y(t) = [1 + \frac{(-itg^{'})^2}{2!} + \frac{(-itg^{'})^4}{4!} + ...]\mathbb{I} + [\frac{(-itg^{'})}{1!} + \frac{(-itg^{'})^3}{3!} + \frac{(-itg^{'})^5}{5!} + ...]M$$

Where;

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & e^{i\theta} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{i\theta} & 0 & e^{i\theta} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{i\theta} & 0 & 0 & 0 & 0 \\ 0 & e^{-i\theta} & 0 & 0 & 0 & 0 & 0 & e^{i\theta} & 0 \\ e^{-i\theta} & 0 & e^{-i\theta} & 0 & 0 & 0 & e^{i\theta} & 0 & e^{i\theta} \\ 0 & e^{-i\theta} & 0 & 0 & 0 & 0 & 0 & e^{i\theta} & 0 \\ 0 & 0 & 0 & 0 & e^{-i\theta} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-i\theta} & 0 & e^{-i\theta} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-i\theta} & 0 & 0 & 0 & 0 \end{bmatrix}$$

(**We can observe that [Y^2, Y^4, Y^6,....] will give Identity matrices whereas [Y^1, Y^3, Y^5,...] will give the same matrix which is given above as M. So we differentiate them in two groups.)

$$\implies U_y(t) = \cos g^{'} t\mathbb{I} - iM \sin g^{'} t$$

$$\Longrightarrow U_y(t) = \cos\frac{gt}{\sqrt{2}}\mathbb{I} - iM\sin\frac{gt}{\sqrt{2}} \tag{50}$$

Now for Bosonic particles, we need to use a 4-qubit system but for implementing a 4-qubit system we must require a 16×16 matrix because any matrix of order $N \times N$ must satisfy the condition $N = 2^n$ (where n= number of qubits). But we can express the above equation in form of a 16×16 matrix (which we have shown in the next sub-section), instead of a 9×9 matrix, by adding 1 diagonally seven times and placing 0 in other positions. In our situation we need only nine of the sixteen 4-qubit states (mentioned in Table (I)) because for the other seven states we will get the same Unitary matrix as result (i.e. without any change). We will use a 4-qubit system to simulate the above system. Therefore, we first note the results we get after operating $U_y(t)$ on different 4-qubit states so that we can go ahead on drawing the quantum circuit for the same.

Qubit states	Results after $\mathrm{U}_y(t) acts$
$\|0000\rangle$	$(\cos\frac{gt}{\sqrt{2}}\|0000\rangle - i\sin\frac{gt}{\sqrt{2}}e^{-i\theta}\|0100\rangle)$
$\|0001\rangle$	$(\cos\frac{gt}{\sqrt{2}}\|0001\rangle - i\sin\frac{gt}{\sqrt{2}}e^{-i\theta}(\|0011\rangle + 0101))$
$\|0010\rangle$	$(\cos\frac{gt}{\sqrt{2}}\|0010\rangle - i\sin\frac{gt}{\sqrt{2}}e^{-i\theta}\|0100\rangle)$
$\|0011\rangle$	$(\cos\frac{gt}{\sqrt{2}}\|0011\rangle - i\sin\frac{gt}{\sqrt{2}}e^{-i\theta}\|0001\rangle - i\sin\frac{gt}{\sqrt{2}}e^{i\theta}\|0111\rangle)$
$\|0100\rangle$	$(\cos\frac{gt}{\sqrt{2}}\|0100\rangle - i\sin\frac{gt}{\sqrt{2}}e^{-i\theta}(\|0110\rangle + \|1000\rangle) - i\sin\frac{gt}{\sqrt{2}}e^{i\theta}(\|0000\rangle + \|0010\rangle))$
$\|0101\rangle$	$(\cos\frac{gt}{\sqrt{2}}\|0101\rangle - i\sin\frac{gt}{\sqrt{2}}e^{-i\theta}\|0111\rangle - i\sin\frac{gt}{\sqrt{2}}e^{i\theta}\|0001\rangle)$
$\|0110\rangle$	$(\cos\frac{gt}{\sqrt{2}}\|0110\rangle - i\sin\frac{gt}{\sqrt{2}}e^{i\theta}\|0100\rangle)$
$\|0111\rangle$	$(\cos\frac{gt}{\sqrt{2}}\|0111\rangle - i\sin\frac{gt}{\sqrt{2}}e^{i\theta}(\|0011\rangle + 0101))$
$\|1000\rangle$	$(\cos\frac{gt}{\sqrt{2}}\|1000\rangle - i\sin\frac{gt}{\sqrt{2}}e^{i\theta}\|0100\rangle)$

TABLE I. Operator $U_y(t)$ acting on Qubit States.

Now, we need to disentangle the final Quantum states after $U_y(t)$ Operator acts on the Qubit states to be able to create the Quantum circuit. So, we can disentangle the final result as:

$$\left(\cos\frac{gt}{\sqrt{2}}0000 - \sin\frac{gt}{\sqrt{2}}e^{-i\theta}0100\right) = 0 \otimes \left(\cos\frac{gt}{\sqrt{2}}0 - \sin\frac{gt}{\sqrt{2}}e^{-i\theta}1\right) \otimes 0 \otimes 0$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$\left(\cos\frac{gt}{\sqrt{2}}0110 - \sin\frac{gt}{\sqrt{2}}e^{i\theta}0100\right) = 0 \otimes 1 \otimes \left(\cos\frac{gt}{\sqrt{2}}1 - \sin\frac{gt}{\sqrt{2}}e^{i\theta}0\right) \otimes 0$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$\left(\cos\frac{gt}{\sqrt{2}}1000 - \sin\frac{gt}{\sqrt{2}}e^{i\theta}0100\right) = \left(\cos\frac{gt}{\sqrt{2}}1 - \sin\frac{gt}{\sqrt{2}}e^{i\theta}0\right) \otimes 1 \otimes 0 \otimes 0$$

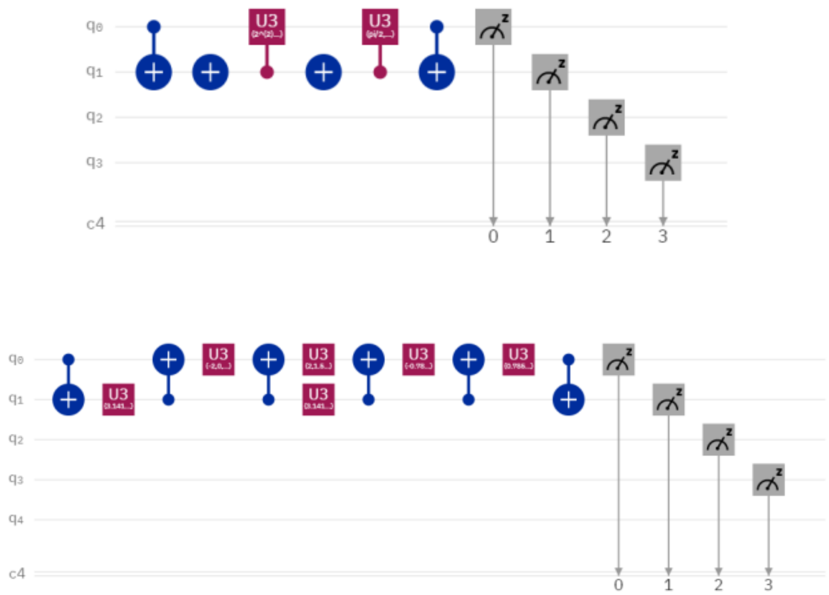

Figure 7. Filter for 1000 in case of $U_y(t)$ Operations

So, in the above segment, we computed the $U_y(t)$ operator and also disentangled the results. A filtered portion of our quantum circuit for the qubit state 1000 is shown in Fig.(7).
Now in order to compute $U_x(t)$ which is equal to e^{-iXt}, we first expand the expression using the Taylor expansion of the exponential function just like we did in earlier case as:

$$U_x(t) = exp(-itX) = \mathbb{I} + \sum_{m=1}^{\infty}(-it)^m\frac{X^m}{m!}$$

$$\Longrightarrow U_x(t) = \mathbb{I} + (-it)^1\frac{X}{1!} + (-it)^2\frac{X^2}{2!} + (-it)^3\frac{X^3}{3!} + (-it)^4\frac{X^4}{4!} + (-it)^5\frac{X^5}{5!} + ...$$

Therefore by using the above equation, we can express $U_x(t)$ in terms of e as:

$U_{\hat{x}}(t)[1,1]$	exp(-($\frac{\omega_0}{2}$)it)
$U_{\hat{x}}(t)[2,2]$	exp(-(0)it)
$U_{\hat{x}}(t)[3,3]$	exp(($\frac{\omega_0}{2}$)it)
$U_{\hat{x}}(t)[4,4]$	exp(-($\omega + \frac{\omega_0}{2}$)it)
$U_{\hat{x}}(t)[5,5]$	exp(-(ω)it)
$U_{\hat{x}}(t)[6,6]$	exp(-($\omega - \frac{\omega_0}{2}$)it)
$U_{\hat{x}}(t)[7,7]$	exp(-($\omega + \frac{\omega_0}{2}$)it)
$U_{\hat{x}}(t)[8,8]$	exp(-(ω)it)
$U_{\hat{x}}(t)[9,9]$	exp(-($\omega - \frac{\omega_0}{2}$)it)

In case of $U_x(t)$ Operator also; we will consider a 16×16 matrix (in place of a 9×9 matrix) because of same reason mentioned before and also we will construct the matrix in the same pattern as mentioned in case of $U_y(t)$ operator. It is easy to observe as X is a diagonal matrix, each diagonal element of $U_x(t)$ makes an exact Taylor expansion of the exponential function
(**The 16×16 matrix for both $U_y(t)$ and $U_x(t)$ operators are mentioned in the next sub-section.)
Again, we operate this operator on different 4-qubits states (in our situation we need only nine of the sixteen 4-qubit states because for the other seven states we will get the same Unitary matrix as result.) and then study the results for the same given in Table(II):
From the above table we can see the effect of $U_x(t)$ operator acting on the different 4-qubit states and we can construct the Quantum circuit for the same. A filtered portion of our quantum circuit for the qubit state 0000 is shown in Fig.(8).
Now, we know how to implement both the parts of our Unitary operator and the complete unitary matrix (16×16) can be implemented by operating both the operations in series. In this way we can easily calculate our Unitary operators for Bosonic system and also simulate the Unitary Operators for a Quantum Harmonic Oscillator. We present its simulation results on IBM quantum computer in the form of graphs later on. Each simulation is carried on IBMQ-qasm Simulator using 8192 shots for better accuracy.

Qubit states	Results after $\mathrm{U}_x(t) acts$
$\lvert 0000\rangle$	$\mathrm{e}^{(-\frac{\omega_0}{2})it}\,\lvert 0000\rangle$
$\lvert 0001\rangle$	$\mathrm{e}^{(0)it}\,\lvert 0000\rangle$
$\lvert 0010\rangle$	$\mathrm{e}^{(\frac{\omega_0}{2})it}\,\lvert 0000\rangle$
$\lvert 0011\rangle$	$\mathrm{e}^{-(\omega+\frac{\omega_0}{2})it}\,\lvert 0011\rangle$
$\lvert 0100\rangle$	$\mathrm{e}^{(-\omega)it}\,\lvert 0100\rangle$
$\lvert 0101\rangle$	$\mathrm{e}^{-(\omega-\frac{\omega_0}{2})it}\,\lvert 0101\rangle$
$\lvert 0110\rangle$	$\mathrm{e}^{-(\omega+\frac{\omega_0}{2})it}\,\lvert 0110\rangle$
$\lvert 0111\rangle$	$\mathrm{e}^{(-\omega)it}\,\lvert 0111\rangle$
$\lvert 1000\rangle$	$\mathrm{e}^{-(\omega-\frac{\omega_0}{2})it}\,\lvert 1000\rangle$

TABLE II. Operator $U_x(t)$ acting on Qubit States.

UNITARY OPERATOR MATRIX REPRESENTATIONS

The 16×16 Matrix representation of the Unitary operators $U_y(t)$ and $U_x(t)$ are:

$$U_y(t) = \begin{bmatrix}
A & 0 & 0 & 0 & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & A & 0 & B & 0 & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & A & 0 & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & C & 0 & A & 0 & 0 & 0 & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C & 0 & C & 0 & A & 0 & B & 0 & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & C & 0 & 0 & 0 & A & 0 & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & C & 0 & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & C & 0 & C & 0 & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & C & 0 & 0 & 0 & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

Where;

$$\text{A} = \cos\left(\frac{gt}{\sqrt{2}}\right) \quad ; \quad \text{B} = -i\sin\left(\frac{gt}{\sqrt{2}}\right)e^{i\theta} \quad \text{and} \quad \text{C} = -i\sin\left(\frac{gt}{\sqrt{2}}\right)e^{-i\theta}$$

Figure 8. Filter for 0000 in case of $U_x(t)$ Operations

$$U_x(t) = \begin{bmatrix}
S & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{S} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & P & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & Q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & R & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & P & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & Q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

Where,

$$\mathrm{P} = e^{-(\omega+\frac{\omega_0}{2})it} \; ; \quad \mathrm{Q} = e^{(-\omega)it} \; ; \quad \mathrm{R} = e^{-(\omega-\frac{\omega_0}{2})it} \; ; \quad \mathrm{S} = e^{(-\frac{\omega_0}{2})it} \quad \text{and} \quad \frac{1}{s} = \frac{1}{e^{(-\frac{\omega_0}{2})it}} = e^{(\frac{\omega_0}{2})it}$$

9. RESULTS

In first part of our paper, we extend the idea of simulation of quantum harmonic oscillator by performing the simulation using higher number of qubits especially 5-qubits and we realized that we are well able to do so using the IBMQ-qasm simulator present on the IBM Experience site. Our initial expectation was that taking higher number of qubits result in larger number of mesh points in the discretized space and hence results are produced with higher degree of accuracy in comparison with the system with lower number of qubits. We come to that result only after comparing data that we got during simulation using 3-qubit and 5-qubit system. The result of simulation of 3-qubit system is shown in Fig.(9).

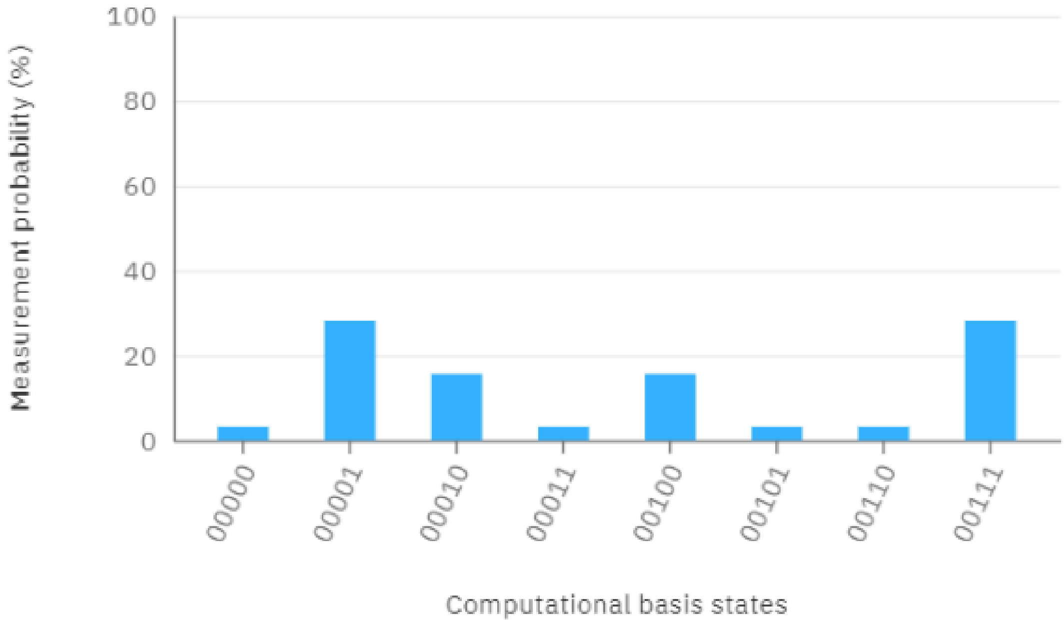

Figure 9. Graph between Measurement Probability vs Computational states in 3-qubit system.

In the final part of our paper, we use the idea of Pauli Matrices Equivalents for Bosonic particles and see the implementation of the equivalent matrices. Then we introduce a coupled Quantum Harmonic Oscillator to the Bosonic system and try to implement its Unitary Operator to the system using our previous section's knowledge and also simulate the Unitary Operators using IBMQ-experience (in 8192 shots for better accuracy). The Results of the simulation are shown in Fig.(10), Fig.(11) and Fig.(12) respectively.

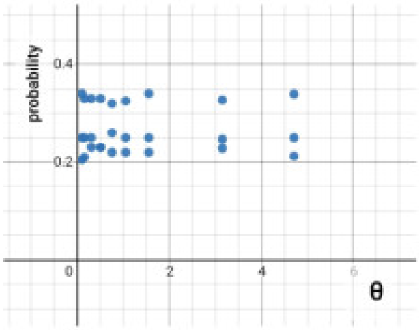

Figure 10. Graph between θ vs Probability

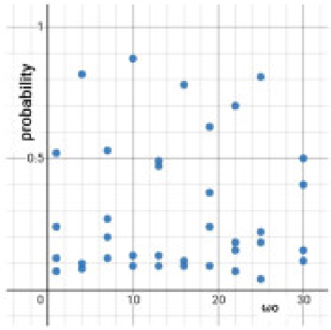

Figure 11. Graph between ω_0 vs Probability

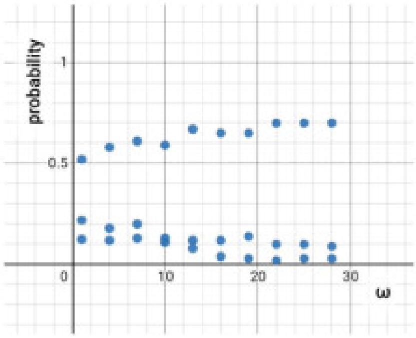

Figure 12. Graph between ω vs Probability

10. CONCLUSION

Now, we understand that the Quantum Harmonic Oscillator **(QHO)** is different from its classical counterpart in many aspects, so oscillation in classical case cannot be pictured in quantum realm. However, there is still something which is measurable and keeps the essence of Quantum Harmonic Oscillator alive. The measurable stuff that we are talking about is none other than the probability amplitudes of the states itself. From the simulations that we obtain on the IBMQ-Experience platform, it can be concluded that the variations of probability amplitudes of the states correspond to the oscillations in quantum sense. Moreover, by performing the simulations in two spacial dimensions, we can state that the results in each dimension is quite independent of the other which was expected intuitively. Clearly, for more accuracy, the number of qubits has to be increased and the generalization section of our work has to be used as it is valid for arbitrary number of qubits. In this project, we visualized the process for simulating a Quantum Harmonic Oscillator **(QHO)**, associated to a Bosonic system, using IBMQ-experience. In our case we derived the Unitary Operators for the **(QHO)** by using the Pauli Matrix equivalents for Bosonic system and after that we associated the usable Quantum States (4-qubit states) with the Unitary Operator (which is in-turn formed by combining the $U_y(t)$ and $U_x(t)$ Operators in series). From the above process we can infer that the Unitary Operator is the sole factor which is necessary for simulating the **(QHO)** and we simulate the system by taking 8192 shots in IBMQ-experience because it will increase the effectiveness of our results and decrease the chance of any error in our simulation.

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