Networks of topological string defects with different Non-Abelian fundamental groups

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Abstract. Topological string defects corresponding to a non-Abelian fundamental group have interesting property of entanglement thus giving rise to the possibility of network formations. We consider some conventional as well as some hypothetical non-Abelian fundamental groups and investigate possibility of different types of networks in each case. The network structures we discuss are one of these three : Hexagonal, Cubic and Diamond lattice.
Keywords: Topological defects, Mean-Abelian fundamental group, String crossing, Biaxial nematics.

1. INTRODUCTION

An ordered media is characterised by an order parameter associated at each point of the system. The collection of all possible values of order parameters gives us an order parameter space. It can be argued on the grounds of topology that certain arrangements of the order parameter produce singularities that cannot be removed by local deformations. Such defects that arise entirely due to topology are called topological defects. These are then categorised into monopole defects, string defects ,etc. Here our primary interest lies in string defects. As we shall see, in certain cases of media(non-Abelian) line defects get entangled i.e. when we cross one string defect around other there remains behind a link joining the two defects at the crossing points. If the topology of the order parameter space (specifically, the structure of its fundamental group) allows it, we can have three or more links at a given point on a string defect giving rise to the possibility of network structures. Almost all the cases that we shall consider here have symmetry groups that have not been observed apart from the case of biaxial nematics (D_2 point group symmetry). Biaxial nematic ordering is expected to arise in certain liquid crystal systems, though it has never been observed as a thermotropic liquid crystal phase. For most of the discussion here, we have followed ref. [1]

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2. CLASSIFYING STRING DEFECTS

2.1 *ORDER PARAMETER SPACE AS COSET SPACE*

Every ordered media has associated with it a function that assigns with each point of space a specific value of an order parameter. Additionally all such order parameter values have associated with them 'a group of transformation G, such that if f_1 and f_2 are any two possible values of the order parameter then there exist a group element g such that $f_2 = gf_1$.' Fixing f, if we define H to be the set of all g such that $gf = f$, then H forms a subgroup of G called as the isotropy group of f. It can be shown that there is an one-one correspondence between elements of order parameter space and elements of coset space G/H giving us a powerful description of the order parameter space.

2.2 *FUNDAMENTAL GROUP*

If we consider loops in physical space then we get corresponding loops in the order parameter space R. If we have a single special value of the order parameter (say x) fixed in those loops then we get based loops. Homotopy classes of such based loops form a group structure. (For this to hold the order parameter space must be path connected.) This is called as the fundamental group of R, or the first order homotopy group $\pi_1(R, x)$.

2.3 *CONJUGACY CLASSES*

The elements of $\pi_1(R, x)$ each depicts a kind of line defect. These essentially should not be able to convert to one another by continuous transformation. But as we have imposed the condition that one point x to be fixed to impose a group structure, we have not exhausted all possibilities of conversion by smooth deformations. The construct of having a fixed point in the loop is artificial and there should indeed be no restriction on such loops apart from the fact that the loops in physical space must surround the defects. For this we invoke a result that links the point based homotopy classes to the 'free homotopy classes'. A loop f at x is freely homotopic to a loop g at y if there is a path isomorphism c taking homotopy class $[f]$ of $\pi_1(R, x)$ into the homotopy class $[g]$ of $\pi_1(R, y)$.' This actually rearranges elements of $\pi_1(R)$ and the sets within which there is shuffling are the conjugacy classes. Thus elements of same conjugacy class can be converted to one another by taking them around another defect, while elements belonging to different conjugacy classes can never be converted to each other by smooth deformations and hence are genuinely topologically distinct defects. Thus a given conjugacy class of the fundamental group describes the set of all the inter-convertible string defects.

2.4 *FUNDAMENTAL THEOREM ON FIRST HOMOTOPY GROUP OF COSET SPACE*

In general, it is very difficult to directly compute $\pi_1(R)$. But a very important result lets us calculate it with much more ease:'If G is a connected and simply connected group, H is any subgroup of

G and H_0 is the set of points in H that are connected to the identity by continuous paths lying in H, then H/H_0 is isomorphic to the fundamental group $\pi_1(G/H)$ i.e. $\pi_1(G/H) \sim H/H_0$.' Thus calculating the fundamental group reduces to calculation of H/H_0.

3. EXAMPLES OF STANDARD AND HYPOTHETICAL FUNDAMENTAL GROUPS

3.1 *ORDINARY SPIN*

This order parameter space consisting of 3 dimensional unit vectors is found in variety of systems with the most widely known being the magnetisation of a ferromagnetic material. Here the group G can be taken to be $SO(3)$. H then is simply $SO(2)$. Thus we have $R = G/H$. But $SO(3)$ is not simply connected debarring us from use of the fundamental theorem. If we take $SU(2)$ then we have $R = SU(2)/U(1)$ where $U(1)$ is the lift of $SO(2)$ in $SU(2)$.But we have $U(1)$ which is connected to identity. Thus, $\pi_1(SU(2)/U(1)) = 0$. Thus there are no topological line defects in such a medium.

3.2 *NEMATIC LIQUID CRYSTALS*

In nematic liquid crystals G can be taken to be $SO(3)$ while H would then be D_∞ as their symmetry group is same as that of cylinder. If D'_∞ is the lift of D_∞ in $SU(2)$, it turns out that D'_∞ has two connected components. So, $\pi_1(SU(2)/D'_\infty) = Z_2$. Thus we have a non trivial defect and the fundamental group of the order parameter space is Abelian.

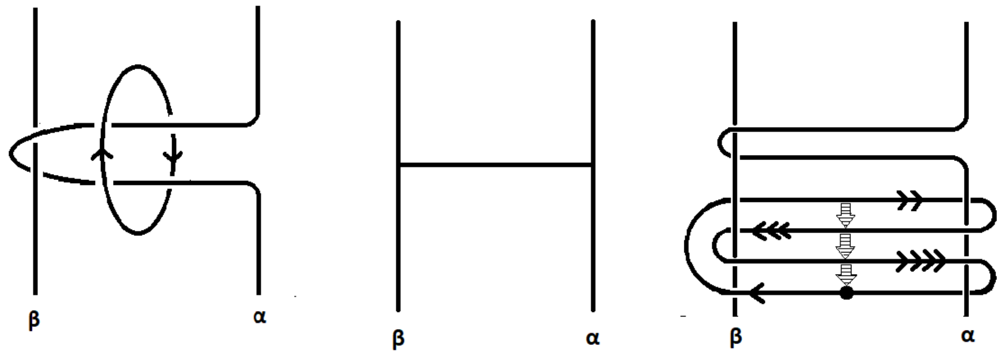
3.3 *BIAXIAL CRYSTALS*

These have a symmetry of rectangular box with point group D_2. Thus order parameter space $R = SO(3)/D_2$. The lift of D_2 in $SU(2)$ is isomorphic to quaternion group, $Q = \{\pm 1, \pm i\sigma_x, \pm i\sigma_y, \pm i\sigma_z\}$, where $\sigma_i' s$ are Pauli matrices. If the group H is discrete then $H/H_0 = H$. Thus we have $\pi_1(SU(2)/Q) = Q$. As Q is non-Abelian, biaxial nematics are thus non-Abelian. These split into five conjugacy classes: $\{1\}, \{-1\}, \{\pm i\sigma_x\}, \{\pm i\sigma_y\}, \{\pm i\sigma_z\}$.

Following two examples of fundamental group have been discussed in ref.[2].

3.4 *TRIANGULAR SYMMETRY*

If the order parameter has a D_3 symmetry group, the order parameter space would then be $R = SO(3)/D_3$. The lift of D_3 in $SU(2)$ is given by $D'_3 = \{\pm 1, \pm\omega, \pm\omega^2, \pm j, \pm j\omega, \pm j\omega^2\}$ where $j = -i\sigma_x$ and $\omega = -\frac{1}{2} - \frac{\sqrt{3}}{2}\sigma_z$. Just like above we conclude $\pi_1(SU(2)/D'_3) = D'_3$. This also turns out to be non-Abelian and has six conjugacy class: $\{1\}, \{-1\}, \{\omega, \omega^2\}, \{-\omega, -\omega^2\}, \{j, j\omega, j\omega^2\}, \{-j, -j\omega, -j\omega^2\}$. Thus we have five different kind of non-trivial defects that cannot inter-convert, see ref.[2] for details.



(a) (b) (c)

Figure 1.: Finding homotopy class of links formed by entangled strings

3.5 *OCTAHEDRAL SYMMETRY*

If order parameter has tetrahedral symmetry group, then $R = SO(3)/T$ where T is the chiral tetrahedral group. The lift of T in $SU(2)$ splits into seven conjugacy classes:$\{1\}, \{-1\}, \{\pm\alpha, \pm\beta, \pm\alpha\beta\}, \{\gamma,$-$\alpha\gamma, \beta\gamma, -\alpha\beta\gamma\}, \{-\gamma, \alpha\gamma, -\beta\gamma, \alpha\beta\gamma\}, \{\gamma^2, \alpha\gamma^2, -\beta\gamma^2, \alpha\beta\gamma^2\}, \{-\gamma^2, -\alpha\gamma^2, \beta\gamma^2, -\alpha\beta\gamma^2\}$, where $\alpha = \frac{1}{\sqrt{3}}\sigma_x + \frac{1}{\sqrt{6}}\sigma_y - \frac{1}{\sqrt{2}}\sigma_z$, $\beta = -\frac{1}{\sqrt{3}}\sigma_x - \frac{1}{\sqrt{6}}\sigma_y - \frac{1}{\sqrt{2}}\sigma_z$ and $\gamma = -\frac{1}{2} - \frac{\sqrt{3}}{2}\sigma_x$, see ref.[2] for details.

4. ENTANGLEMENT OF LINE DEFECTS

4.1 *NON-ABELIAN MEDIA AND ENTANGLEMENT*

Let us consider a simple exercise for two line defects that aren't exactly coplanar. If we try to cross one string defect α around the other defect β then we arrive at a configuration similar to figure 1(a). We can pinch the overlapping parts to get a construct like figure 1(b). It may so happen that this pinched defect may vanish then we simply recover our original configuration. But if it happens to be non-trivial then we are left with a link between the two line defects and arrive at the so called 'entanglement' of line defects.

To determine the homotopy class of the link consider a loop around it and deform it to arrive at a configuration similar to figure 1(c). We conclude that the class of the link is $\beta \circ \alpha \circ \beta^{-1} \circ \alpha^{-1}$. A similar exercise (in figure 2) suggests that the upper half of α changes to $\beta \circ \alpha\beta^{-1}$. The link is trivial if $\alpha \circ \beta = \beta \circ \alpha$ i.e. medium is Abelian. Thus we can only have non-trivial links in case of non-Abelian media. (When we say the medium is Abelian or non-Abelian, we mean that the fundamental group of the corresponding order parameter space is Abelian or non-abelain respectively.)

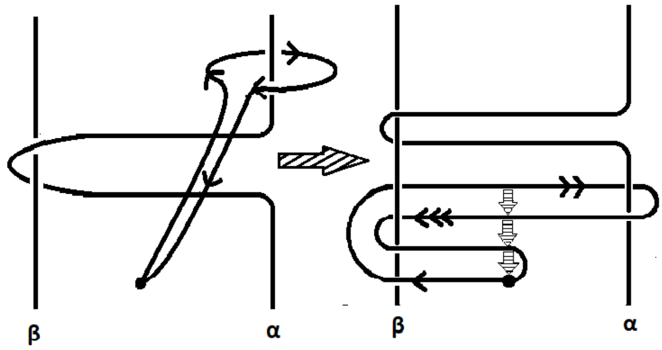


Figure 2.: Finding homotopy class of string after crossing the other string

4.2 *PROPERTY OF THE CLASS THE LINKS BELONG TO*

If σ is a link i.e. $\exists\ \alpha$ and β such that $\alpha \circ \beta \circ \alpha^{-1} \circ \beta^{-1} = \sigma$, then all the elements of the conjugacy class that σ belongs to are valid links too .To frame the claim mathematically: If $\sigma' \in A$ where A is the conjugacy class of σ, then $\exists\ \alpha'$ and β' such that $\alpha' \circ \beta' \circ \alpha'^{-1} \circ \beta'^{-}1 = \sigma'$.

The above statement is true because: $\gamma \circ \{\alpha \circ \beta \circ \alpha^{-1} \circ \beta^{-}1\} \circ \gamma^{-1} = \{\gamma \circ \alpha \circ \gamma^{-1}\} \circ \{\gamma \circ \beta \circ \gamma^{-1}\} \circ \{\gamma \circ \alpha^{-1} \circ \gamma^{-1}\} \circ \{\gamma \circ \beta^{-1} \circ \gamma^{-1}\} = \alpha' \circ \beta' \circ \alpha'^{-1} \circ \beta'^{-1}$

Another important observation about the conjugacy classes that such links may belong to is the fact that the conjugacy class must contain inverses of all of its elements i.e. if $\alpha \in A \implies \alpha^{-1} \in A$ where A is the conjugacy class. This is an observation without proof. This seems to be a necessary condition but not a sufficient one for a conjugacy class to produce links.

5. LINKS POSSIBLE IN VARIOUS NON-ABELIAN MEDIA

If we have a non-Abelian media then we are guaranteed to get atleast one conjugacy class of links (simply because if no such link exist then it means that the group is Abelian). Thus even in case of the most trivial non-Abelian media we are guaranteed to get links connecting various line defects. The following text describes the kind of links as well as when they form in the specific non-Abelian media. We shall use $\longrightarrow$ to mean first string crosses the other.

5.1 *Biaxial Nematic Liquid Crystal*

From multiplication table of the quaternion group, $\beta \circ \alpha \circ \beta^{-1} \circ \alpha^{-1}$ can only be either 1 or -1. The non-trivial -1 link forms when $\pm\sigma_i \longrightarrow \pm\sigma_j$ such that $i \neq j$ and the α (i.e. the one that crosses) changes sign. Thus we have only one kind of link in biaxial nematic.

5.2 *Triangular Symmetry*

Here we have a bit more diversity then previous case as links can be trivial or ω or ω^2. We have ω link if $\pm\omega \longrightarrow \{j, j\omega, j\omega^2\}, \{-j, -j\omega, -j\omega^2\}$; $\{j, j\omega, j\omega^2\}, \{-j, -j\omega, -j\omega^2\} \longrightarrow \pm\omega^2$; $\pm j \longrightarrow \pm j\omega$; $\pm j\omega \longrightarrow \pm j\omega^2$ or $\pm j\omega^2 \longrightarrow \pm j$. To get a ω^2 link we can cross the following: $\pm\omega^2 \longrightarrow \{j, j\omega, j\omega^2\}, \{-j, -j\omega, -j\omega^2\}$; $\{j, j\omega, j\omega^2\}, \{-j, -j\omega, -j\omega^2\} \longrightarrow \pm\omega$; $\pm j \longrightarrow \pm j\omega^2$; $\pm j\omega^2 \longrightarrow \pm j\omega$ or $\pm j\omega \longrightarrow \pm j$. The upper half of the crossing link can be computed by calculating $\beta\alpha\beta^{-1}$.

5.3 *Octahedral Symmetry*

This fundamental group has got a very complicated crossing and varied classes of links. The links can belong to either $\{-1\}$ or $\{\pm\alpha, \pm\beta, \pm\alpha\beta\}$. The α and β in this case is very complicated. We can have -1 only when $\{\pm\alpha, \pm\beta, \pm\alpha\beta\}$ crosses members of itself in certain cases. The other links form when other conjugacy classes cross among each other where in some cases we get trivial links. This increases our pool of distinct classes of links to seven. As we shall see it provides a good set to form $3-$dimensional lattice of networks.

6. NETWORKS POSSIBLE IN VARIOUS MEDIA

The previous section gives us a possibility of forming networks of defects. This section gives a detailed understanding of the types as well as the expected stability of different structures.

6.1 *STITCHING PATTERN*

The way in which we 'stitch' the defects affects the overall stability of the network that we construct. By 'stitching' we mean the manner and order in which the links are formed and stacked. The way we do it is analogous to how we weaves a piece of cloth and the whole arrangement of thread holds its shape and not separate into a mass of independent threads. As we shall see the networks that we construct 'remembers the way it was stitched and thus maintains its overall structure and this stability is topological i.e. it's stability does not depend on the energetics rather is intrinsic to the medium itself. So it is important to define a proper way of stitching that if constructed for threads would maintain the structure.

6.2 *NON-CROSSING OF LINKS*

The links are formed when one string crosses the other. Thus links are independent entity. When stacked upon each other we get numerous links at a particular node. It is important to note that the links should be such that they can't bypass each other and always maintain the order in which they we stitched. In most cases the above statement is true. But there exist cases where the above

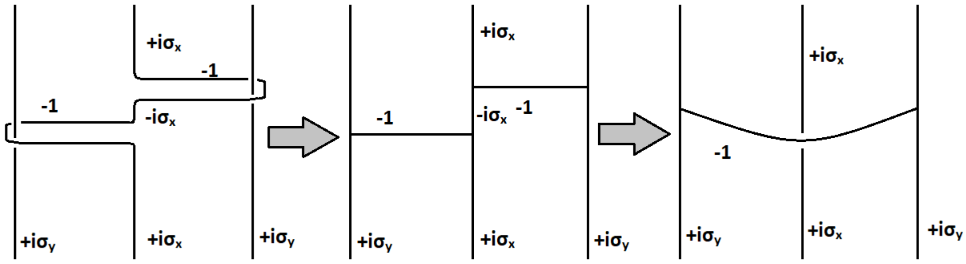


Figure 3.: Possible untangling of links

statement is false. So while looking at networks of defects it is important that links don't bypass each other so that it maintains the structure the way these were put together.

6.3 *POSSIBLE LATTICE OF DEFECTS*

Keeping in mind the above points and the fact that we should avoid stacking up same classes of defects consecutively we can have the following basic lattices.

6.3.1 *HEXAGONAL LATTICE*

Let us consider the simplest of all the non-Abelian fundamental group, biaxial nematic. As we have only one class of link i.e $\{-1\}$ possible so it rules out the possibility of more than one branch at a particular node made by a single string.

As shown in figure 3, there is a possibility of the links separating out if we have two branch originating from the same string at a node. So it is wise to opt for a safe one branch network. If 'tension' remains constant then the most stable network is that of a 2-D hexagonal lattice.

It is important to note the stitching pattern as this is an important factor in deciding stability of the network. The scheme is shown in figure 4.

In figure 4, each branch from a string is sandwiched between two branch of the adjacent string . So, if we try to separate out a link by joining two -1 links as done previously we shall end up with a point where we have two distinct class of defects in point based homotopy group joining which contradicts the very meaning of being distinct class in a point based homotopy group. As we can see in figure 5, we have a continuity problem for the string that changes from $-i\sigma_x$ to $+i\sigma_x$.

As shown in figure 4, the consecutive string defects belong to different conjugacy classes. In fact our only requirement in this case is to have no two adjacent string defects belonging to same conjugacy class and the choice can be completely random which still remains topologically stable. Thus we can have infinite such arrangements.

We can rearrange the stitching in figure 4 to get a 2-D square lattice as in figure 6.

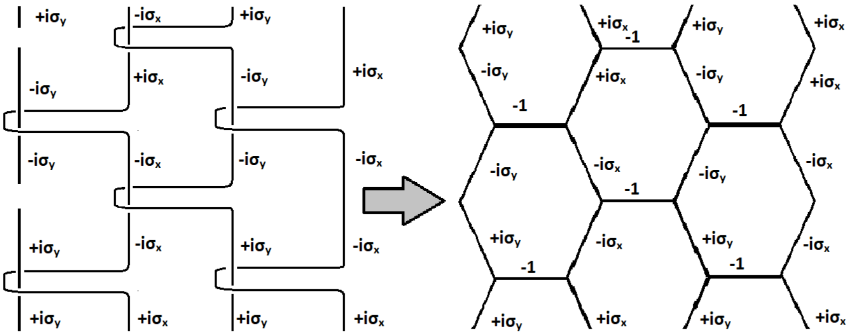


Figure 4.: Stitching pattern for 2-D hexagonal lattice

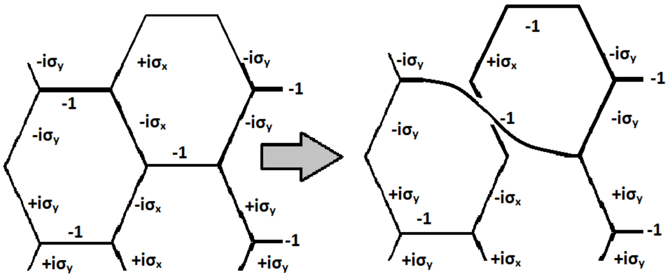


Figure 5.: Infeasibility of untangling the links

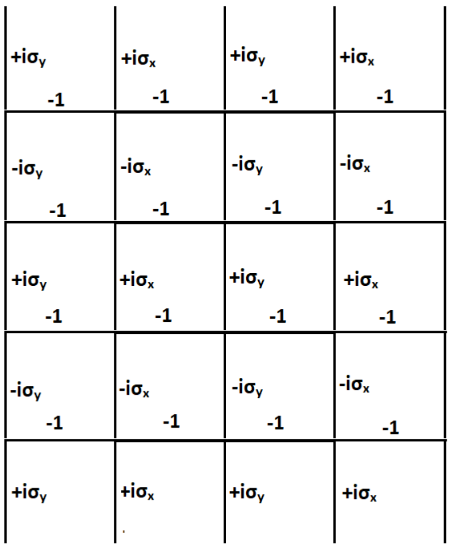


Figure 6.: 2-D square lattice with biaxial nematic

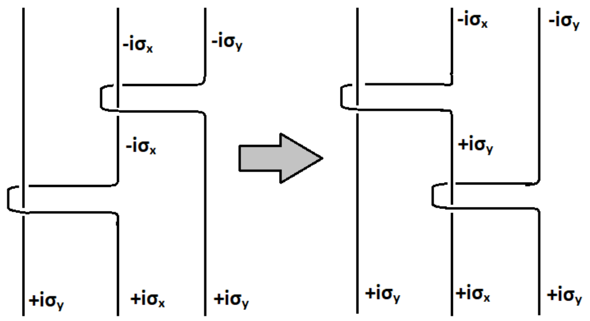


Figure 7.: Sliding of links

As the stitching which forms the cubic lattice is topologically stable so is the lattice formed by sliding links.

However links here can cross each other as in figure 7. Thus we have some kind of mobility in the stitching we did for biaxial nematic.

This can be overcome and a perfectly non-mobile topologically stable stitching can be obtained if we use fundamental group of triangular or octahedral symmetry.(Refer figure 8)

Here in figure 8 the string 1 shows a periodic change from j to $j\omega$ to $j\omega^2$ while string 2 alternates between ω and ω^2. As can be verified, it overcomes the previous problems of link mobility.

The hexagonal crossing can have a variants too.

In pattern 1 (refer figure 9), we have a general way of building hexagonal lattice. We trace a line in the hexagonal lattice such that if we add up the angle we turn its absolute value should not exceed 180^0. This allows us to stack up the same pattern on top of one another. Pattern 2 uses the same principle while making use of the 180^0 change in direction of defects. In pattern 3 we have the angle of turns add up to be greater than 180^0 so that the next defect covers more space and thus we have a family of defects of increasing width. As can be noted one string must cross itself for it to form the lattice but one class commutes with itself so that we have no links along their contact barring the formation of complete lattice.

6.3.2 *CUBIC LATTICE*

We directly make use of the largest non-Abelian fundamental group at our disposal to show the feasibility of cubic lattice. Our requirement for stability of the structure is that neither stitching should not crumble upon itself nor should the links be mobile as noted earlier. Avoiding those two points ensures topological stability.

In case of tetrahedral symmetry, we have two different two different conjugacy class of links as well as seven different classes of them. For the sake of simplicity, we shall make use of only one conjugacy class i.e. $\{\pm\alpha, \pm\beta, \pm\alpha\beta\}$. Thus we have more freedom of stacking up links. We shall

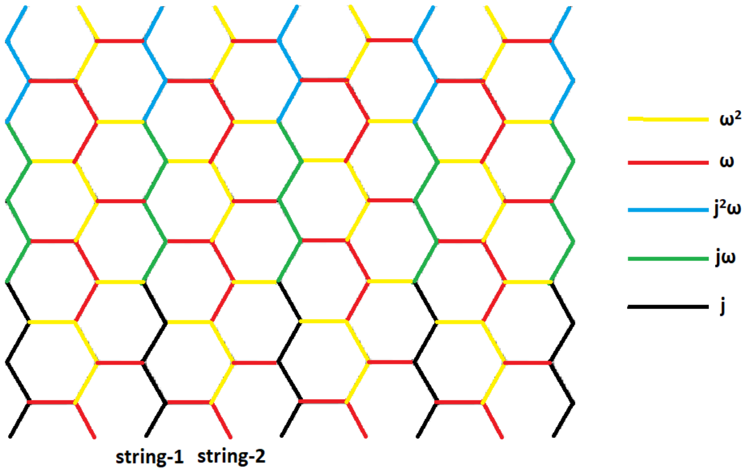


Figure 8.: 2-D hexagonal lattice formed with fundamental group of triangular symmetry

adopt a similar stitching as in hexagonal lattice(refer figure 10).

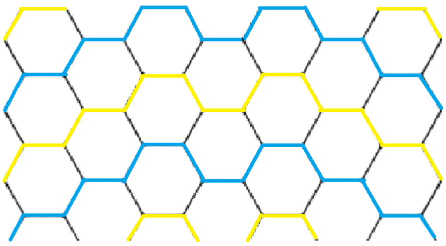
Figure 11 shows two layers of such a structure. These two layers sandwich each other and form the whole of the cubic lattice as shown in Figure 14. Each node has different links. As can be verified the links can neither be untangled nor crossed across each other. So it's a topologically stable structure. We can choose elements of other conjugacy classes too to get a similar structure.

6.3.3 *DIAMOND LATTICE*

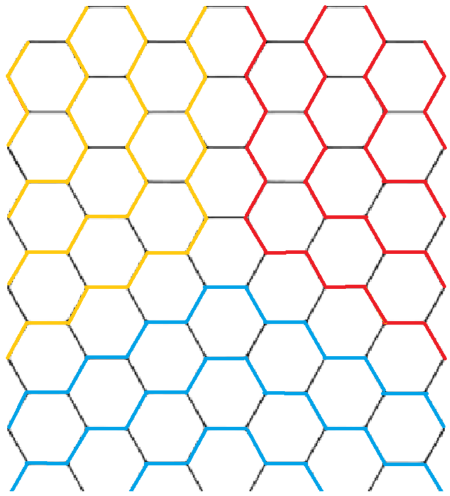
The stitching used in forming cubic lattice can be easily changed to a diamond lattice (refer figure 13). The model shown in figure 13 is made using wires and colour coded using tapes to identify various classes of string defects. This is similar with what we did with 2-D hexagonal to square lattice.

6.4 *POSSIBILITY OF OTHER NETWORKS*

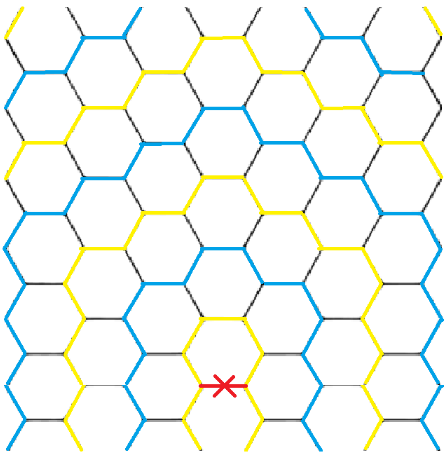
We have only explored the lattices which would have a constant tension throughout. But the number of lattice possible is theoretically infinite, depending on differences in tensions of different strings. But the fact that we have to trace out lines apart from the links to completely make that structure should be kept in mind. Apart from standard lattice we can do a bit more with the fundamental group of the tetrahedral symmetry. If we choose our strings to be γ and $\alpha\beta\gamma^2$ then we can build bilayers using the stitching as used in cubic lattice because these change into $-\alpha\beta\gamma$ and $\alpha\gamma^2$ respectively (refer figure 14). Then we can stitch in an inverted way below also to get a topologically stable structure. We can take various other elements to get the required result as in other cases.



(a) *Pattern* 1



(b) *Pattern* 2



(c) *Pattern* 3

Figure 9.: Variants of 2-D hexagonal lattice

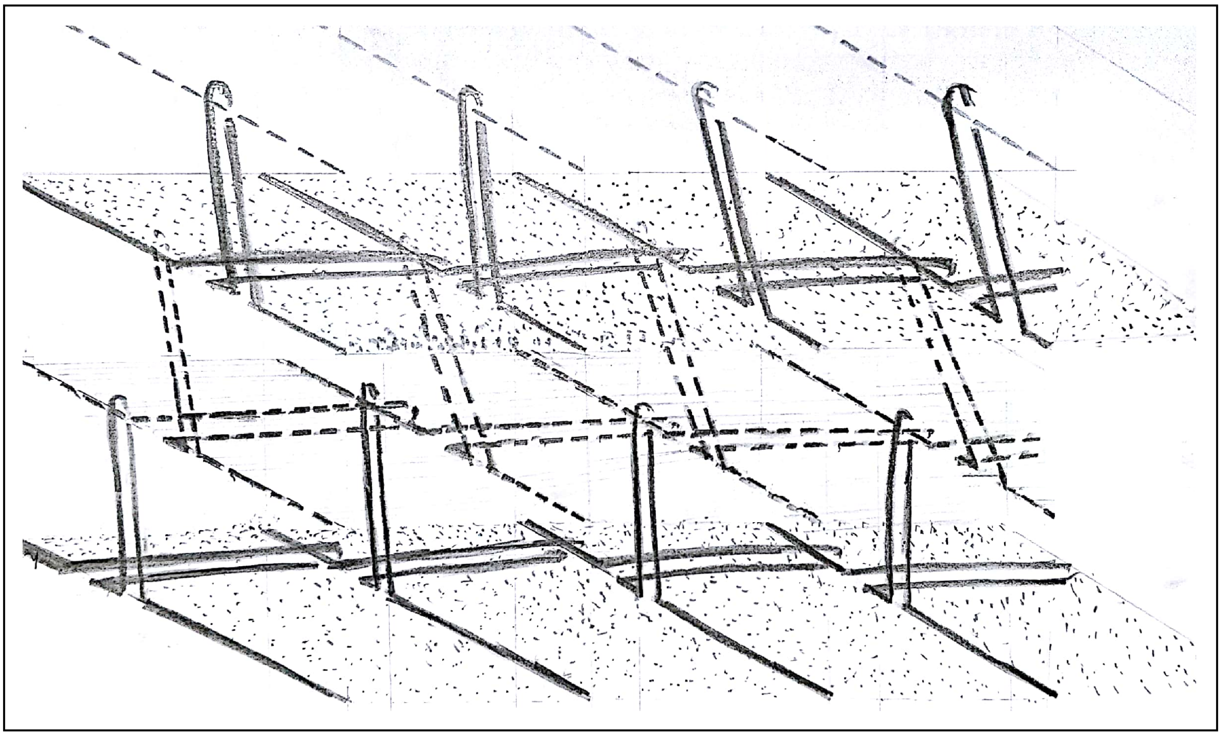
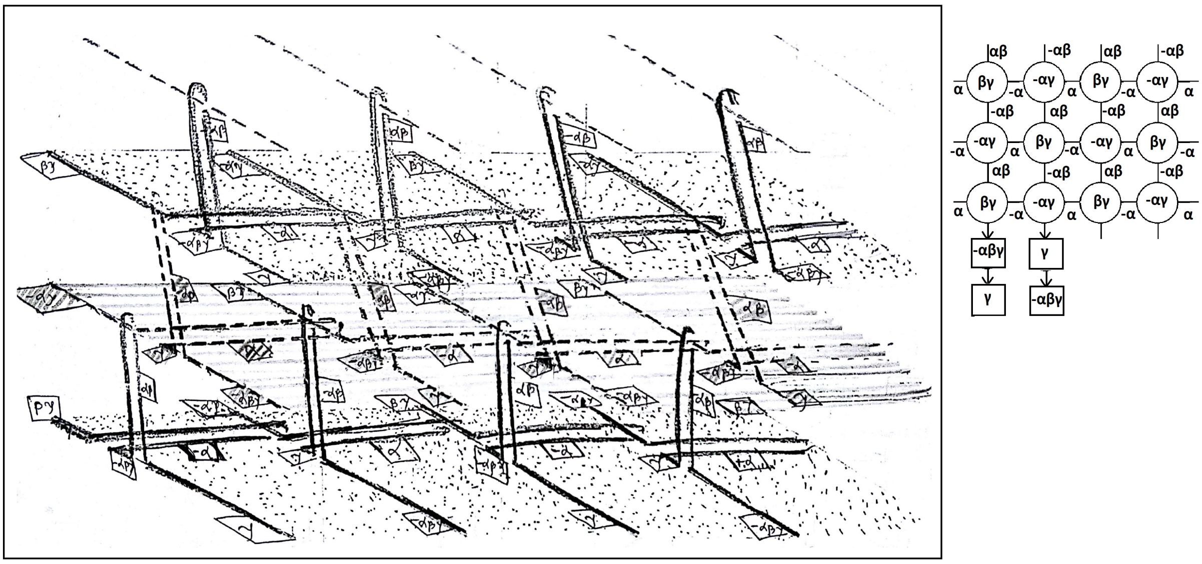
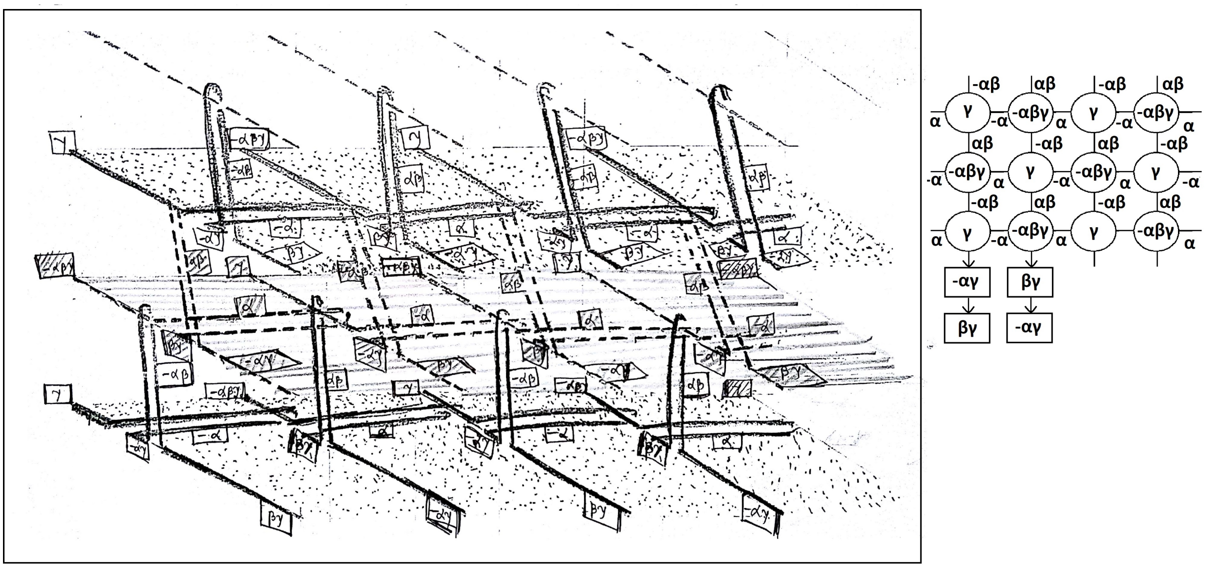


Figure 10.: Stitching patten for cubic lattice



(a) *Layer* 1



(b) *Layer* 2

Figure 11.: Layers of a cubic lattice

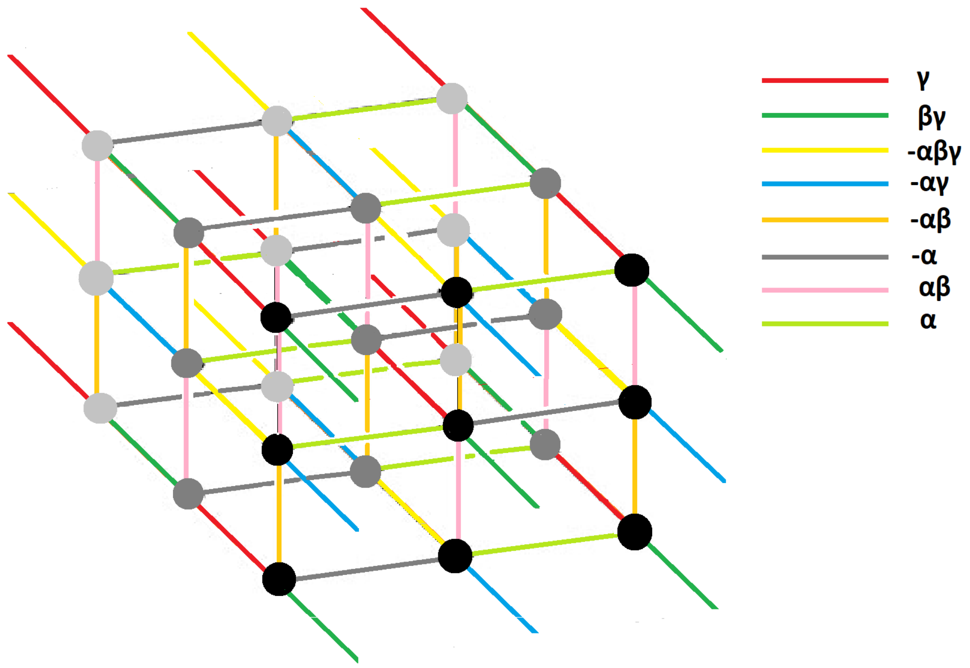


Figure 12.: Layers of a cubic lattice

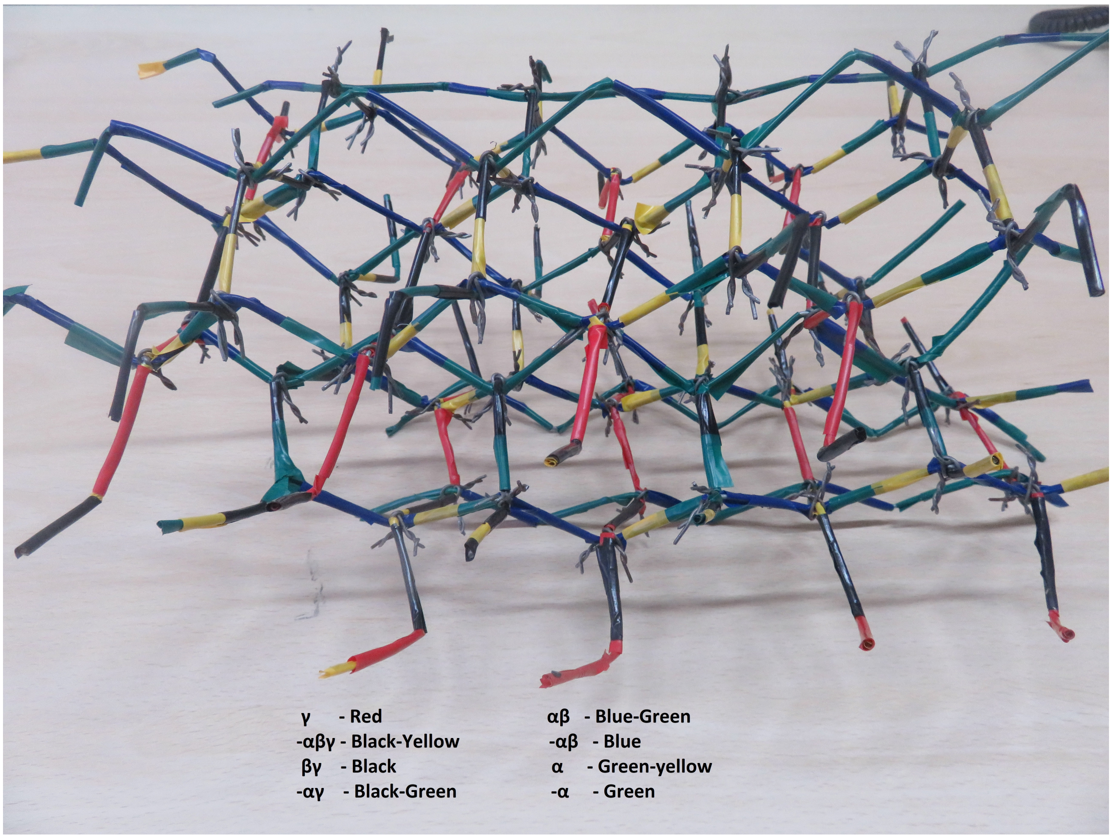


Figure 13.: Diamond lattice

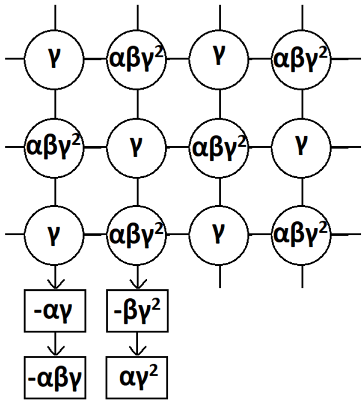


Figure 14.: Block diagram for bilayer

If we have fundamental groups of higher order symmetry then we can build more complicated nodes. Here in the example we used for cubic and diamond lattice have only two layers that repeat and each layer has only two different kinds of sting. We can in fact use more number of strings of different classes and in turn get periodicity of varying order.

7. CONCLUSION

To conclude, we have demonstrated topological stability of network structures formed from string defects in an non-Abelian medium. The diversity of classes in the fundamental group largely affects the variety of networks possible. As has been noted above, ensuring the stability of the stitching pattern used to construct a network ensures the network's stability. An ordered media such as liquid crystal or ferromagnetic materials are largely affected by the presence or absence of defects in them. In case of biaxial nematic liquid crystals or other symmetry groups (if discovered) could allow for more control over the distribution of defects by formation of networks of them affecting the material properties. Thus in those cases such networks might play an crucial role in material science.

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