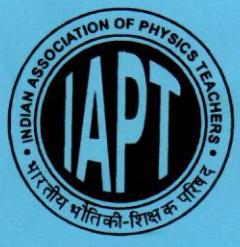
ISSN -2319-3166

STUDENT JOURNAL OF PHYSICS

Volume: 4 Number: 5 Oct.-Dec. 2012



INDIAN ASSOCIATION OF PHYSICS TEACHERS

STUDENT JOURNAL OF PHYSICS

This is a quarterly journal published by Indian Association of Physics Teachers. It publishes research articles contributed by Under Graduate and Post Graduate students of colleges, universities and similar teaching institutions, as principal authors.

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TURNING POINTS

A Brief Introduction to Quantum Computation

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Communicated by: D.P. Roy

A brief introduction to the principle behind realization of a quantum computer is presented. After a review of some concepts Quantum Mechanics that are useful in understanding the working of a quantum computer, we look at the various components and processes that go into building a quantum computer. A quantum algorithms for database search is described. Finally, we discuss the criteria for devising a practical quantum computer and the present status of such a device. [1]

1. INTRODUCTION

One of the technological advances that had a profound influence on human progress during the last century was the design of a digital computer. From the limited objective of providing a fast tool for scientific computation with possible technological applications in areas such as space science and meteorology, computers have come to influence our day to day lives through their use in diverse areas, such as, business finance, communication technology (E-mail, mobile telephony) , sound and imaging (DVD players, video games) etc. At the core of these applications is the concept of splitting individual chunk of information (data) into bits which can be stored and manipulated through electronic circuitry. The sequence according to which the bits are to be manipulated in order to achieve the desired objective is known as an algorithm. Behind the successful implementation of every algorithm is the concept of a Turing machine, introduced by the mathematician Alan Turing [2], which consists of three basic elements, viz., (i) a tape consisting of cells each of which can store one bit of information, (ii) a control unit, which has a finite number of states, including a halt state to signify termination of computation and (iii) a read/write head. Though this basic computer is capable of storing and processing a limited amount of information at a time, it has, in principle an unlimited amount of time in which to achieve its goal. The celebrated Church- Turing thesis [3] states that every computable function can be computed effectively by a finite number of steps of a Turing machine.

Ever since the first digital computer was invented, efforts for making faster and faster computers which can store more and more memory have been made. The emergence of semiconductor devices saw an exponential increase in both the processing speed and memory, which were further enhanced by the development of integrated circuits. The co-founder of Intel Corporation, Gordon Moore is credited with predicting the pace of semiconductor technology through "Moore's law" which states that the number of transistors and resistors on a chip doubles every 18 months. Though Moore corrected himself in 1975 and made the doubling period as 24 months instead of 18, a consequence of the dense packing of components on a chip is that as the miniaturization of computing devices continues, macroscopic physics will no longer determine the behavior of the processors; they will instead be determined by quantum laws. It is even conceivable that a single electron may store one bit of information. It may be remarked that though the physics of semiconductor devices on which the present day processors are based can only be satisfactorily explained from a quantum mechanical viewpoint, they are not *quantum* in the sense that information storage and processing in such computers (which we shall designate *classical computers*) are based on a macroscopic two level systems.

Other than the fact that devising a computing machine whose working is based on microscopic laws is an intellectually challenging task, a quantum computer has several advantages over a conventional (classical) computer. The power of a quantum computer is based on two properties of a quantum state, viz., superposition and entanglement. While the state of a classical bit is deterministic, i.e., it is either in the state 0 or in the state 1, a quantum bit (called a qubit) can be simultaneously in both the states. In general, a quantum system can be in a superposition of different states at the same time. Further, since these states evolve linearly with time, all the components of a superposed state are processed simultaneously, giving an unprecedented parallelism which cannot be matched by a classical computer.

In the theory of computation, the difficulty level of a problem is determined by the time an algorithm takes to obtain a solution. For a given input size n, if the time taken $m(t)$ does not exceed that computed by a polynomial of degree k, i.e. if $m(t) \sim O(n^k)$, where k is a constant, the problem is said to be *easy* and solvable in polynomial time whereas a *difficult* , exponential time algorithm is one where the time taken is $O(k^n)$, where $k > 1$. One of the important applications of this property is that a quantum computer may be in a position to solve certain problems in polynomial time which could not hitherto be solved by a classical algorithm. An example of such a problem is Shor's factorization of a composite number [4].

The second property of a quantum state is that the states of two or more particles of a composite system can be so entwined so that if any of the constituent particles is disturbed, the other constituents are also influenced by it. The entanglement of the constituent particles persist even when they are separated by space-like distances. This is in apparent violation of special theory of relativity because when two objects are separated by space-like intervals, no information can reach either of the bodies due to an event happening in the other body unless information can travel with a speed greater than the speed of light in vacuum. This has been termed as "spooky action at a distance" and is of use in applications such as quantum teleportation and cryptography. Feynmann [5] was the first to point out that since the physical world is quantum mechanical, simulating laws of physics using

a classical computer leads to an exponential slowing down in the processing speed of a computation. What exactly is the role of entanglement in increasing computational speed has been a matter of much debate and controversy. It has been suggested [6] that the success of Shor's algorithm is primarily due to the large entanglement of the quantum register.

David Deutsch [7] generalized the Church-Turing thesis to establish what can be called a universal quantum Turing machine. Though it was not able to simulate the machine in all cases in polynomial time, Deutsch's algorithm was faster than the corresponding classical algorithm. Grover [8] designed an algorithm which aims at searching for a particular item from a database of N entries. A classical search algorithm requires $O(N)$ searches whereas Grover's algorithm achieves this task in $O(\sqrt{N})$ steps. Even this quadratic speeding up can be of substantial interest. For instance, consider a database containing entries of the entire one billion population of India. To search for a particular entry in this database (the so called needle in the haystack) at a processing speed of 1000 searches per second would take 10 days of continuous processor time. Grover algorithm would complete this task in about five minutes.

In this review we will introduce the readers to some of these interesting applications of quantum computation. In Section 2 we will review some basic concepts of quantum mechanics which will be useful in understanding the working of a quantum computer. In Section 3, we discuss Grover's search algorithm. In Section 4, we will comment on practical realization of a quantum computer.

2. PRELIMINARY CONCEPTS

2.1. Quantum Mechanics the Copenhagen interpretation

In this section we assume the familiarity of the reader with undergraduate level quantum mechanics. We will highlight some salient features of what has come to be known as the Copenhagen interpretation of quantum mechanics, primarily due to Niels Bohr, but contributed in good measure by Heisenberg, Max Born, Dirac and many others.

In classical physics, the state of a system (which could be described by dynamical quantities such as position, momentum etc.) is determined by its state at an earlier time and through a deterministic set of laws (e.g., Newton's laws) by which the system evolves with time. Thus if we know the state of a system at the present time, we can, by solving the equation of motion, determine its system at any time in the past or in the future. Further, the state of the system is observer independent, i.e., a state is intrinsic to the physical system and is not influenced by the fact whether the state is being observed or not. The accuracy of the measurement is only limited by the accuracy of the measuring apparatus which, in principle, can be made indefinitely accurate. Quantum mechanics, however, imposes a more fundamental limitation on the outcome of a measurement.

While Heisenberg's uncertainty principle and Bohr's principle of complementarity are essential ingredients of the Copenhagen interpretation, here we are mostly concerned with an aspect of the quantum postulates, known as the *collapse of the wave function*. In quantum mechanics, the state of a system is described by a state vector in a linear vector space known as the Hilbert space. The

more familiar wave function is the projection of this vector on to the position space. The state vector evolves with time according to Schrödinger equation. To every physical observable, there corresponds an operator which is Hermitian. This operator acts on the state of the system changing it to a new state. In general, a quantum system exists as a linear superposition of component states, the weight of a given component in the superposition being known as its amplitude. As long as a state is left undisturbed, it evolves with time as per Schrödinger equation. If, on the other hand, an observation is made of a physical observable, it would "collapse" into one of the eigenstates of the operator corresponding to the observable. While the original state was a superposition of various eigenstates of the operator, the probability with which a particular eigenstate would be found as a result of measurement is proportional to the square of the amplitude of the eigenstate in question in the state immediately before measurement. Thus, unlike the situation in classical physics, a measurement of a quantum system does not provide information about the state of the system, but does so only about the state to which it has collapsed.

To illustrate the above, let us consider a quantum system which can exist in one of the two states $\mid 0\rangle$ and$\mid 1\rangle$. These states could represent, for instance, the state of polarization (vertical or horizontal) of a photon or the spin projection of an electron (up or down) or the ground state and the excited of an atom. An arbitrary state $\mid \psi\rangle$ of the system is given by $\mid \psi\rangle = \alpha \mid 0\rangle + \beta \mid 1\rangle$, where α and β are the probability amplitudes of the states $\mid 0\rangle$ and$\mid 1\rangle$ respectively. If the state is normalized, the probability sum demands $\mid \alpha \mid^2 + \mid \beta \mid^2 = 1$. One can easily generalize the situation to the case where the state vector is expressed in an n component basis so that $\mid \psi\rangle = \sum_{i=0}^{n-1} \alpha_i \mid i\rangle$, where the amplitudes α_i satisfy the normalization constraint $\sum_i \mid \alpha_i \mid^2 = 1$. As the system evolves, the state will change with time but the normalization constraint has to remain satisfied. Operators which can act on the state vector while preserving the norm are known as unitary operators and will be denoted by U. A quantum computer works with logic gates which consist of unitary operations.

2.2. Reversibility

An advantage of working with unitary gates is that quantum computing is intrinsically reversible. A gate is said to be reversible if it is possible to retrieve the input from the output by simply reversing the sequence of operations which yielded the output starting with the input. Gates used in classical computation are, in general, not reversible. For instance, an AND gate is not reversible because, given a 0 output, it is not possible to determine which of the three pairs of input, viz., (0,0), (0,1) or (1,0) had yielded this result. Thus the operation of the gate has resulted in erasing one bit of information. Landauer [9] had shown that each time a single bit of memory is erased, a minimum amount $k_B \ln 2$ of energy gets dissipated into the environment. This is known as Landauer's principle. If one could carry out computation without erasing data, we would generate less heat and hence have a more energy friendly device. It is possible to design reversible gates and carry out computation only using such gates. Charles Bennett [10], has shown that it would be possible to make a classical computer reversible by having it save all the information on a blank tape it would otherwise throw away. This would, however, increase the memory requirement of a computer significantly.

The two level state described above can be used to define a quantum bit, known as qubit, as distinct from a classical bit or a *cbit*. For instance, in a classical computer, the voltage between the plates of a capacitor could represent a cbit, with a charged capacitor representing the bit 1 while when the same capacitor is discharged it could represent the cbit 0. Unlike a classical bit which can be either in the state $\mid 0\rangle$ or in the state $\mid 1\rangle$ at a given time (i.e., either the capacitor is charged or discharged), a qubit lies in a vector space parameterized by α and β which can take infinitely many complex values. A qubit in superposition is in both of the states and at the same time. These qubits form the basic unit of quantum computing and quantum information.

One needs to understand this strange concepts of being in two states at the same time. The following experimental situation illustrates this concept.

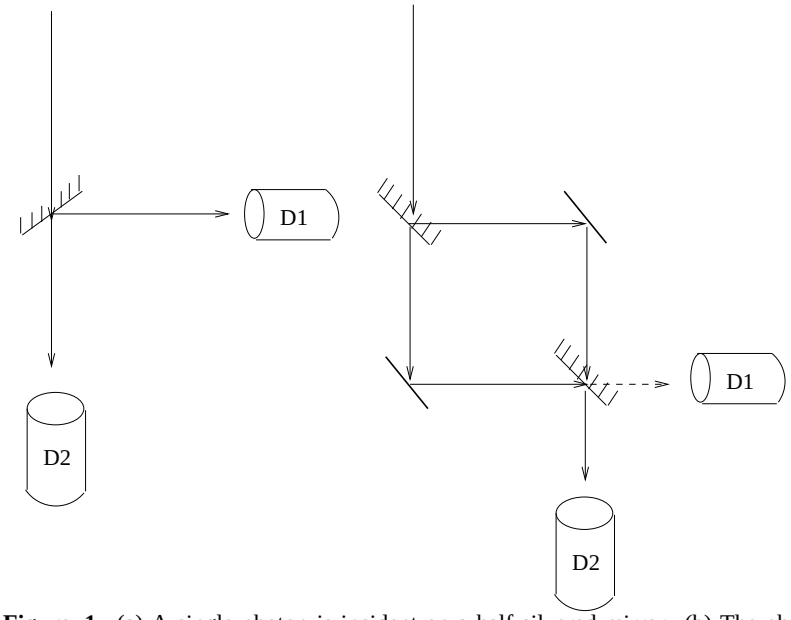

Figure 1. (a) A single photon is incident on a half silvered mirror. (b) The photon undergoes multiple reflections and transmission through the mirror arrangements.

In figure (1a), a single photon is incident on a half silvered mirror. The photon is detected either by the detector D1 or by the detector D2 with equal probability. In figure (1b), the photon first falls on the half silvered mirror and can apparently take either the reflected path or the transmitted path. On both these paths a full silvered mirror is placed which acts as a reflector. A second half silvered mirror allows the photon to reach either of the two detectors. One would expect that the photon, in this case will be detected either by D1 or by D2 with 50% probability. It is, however, found that the photon is detected with 100% probability only by the detector D2 and never by D1. Clearly, in this case the photon must have made up its mind to reach only D2. This can happen if photon has taken both the paths and arrived at the second half silvered mirror to combine constructively for D2 and

destructively for D1. These paths must also have maintained phase correlation to have been able to interfere, i.e., photon must have been in a coherent superposition of being in both the reflected beam and the transmitted beam.

In order that qubits are useful in quantum computation, we need to have states belonging to a composite system. If the composite system consists of several isolated subsystems, the state of the system is simply a direct product of the states of the constituents. One can then keep track of the qubit of each subsystem. A two qubit state $\mid 01\rangle$ denotes the state of a system whose first particle is in a state with qubit $\mid 0\rangle$ and the second in the state with qubit $\mid 1\rangle$. A not so explicit but still identifiable as two separate qubit state is $(1/\sqrt{2})(\mid 00\rangle + \mid 01\rangle)$ which is a product of the first particle in the state $\mid 0\rangle$ and the second in the state $(1/\sqrt{2})(\mid 0\rangle + \mid 1\rangle)$. If one considers interacting systems, it may no longer be possible to factorize the state in this manner. When this happens, we say that the qubits are entangled, a term apparently coined by Schrödinger. The simplest example of an entangled system is a singlet state arising from two spin half particles. The state of this system $1/\sqrt{2}(\mid 01\rangle - \mid 10\rangle)$ cannot be expressed as a product of the states belonging to the individual particles. If one were to measure the state of the first particle and find that it has collapsed to a state $\mid 0\rangle$, the second particle would have instantaneously collapsed to the state $\mid 1\rangle$, no matter how far removed it was from the first particle.

One can use these to generate quantum registers. For instance, a 3 qubit quantum register containing an equally superposition of states is represented by

$$\mid \psi\rangle = \frac{1}{\sqrt{8}}[\mid 000\rangle + \mid 001\rangle + \mid 010\rangle + \mid 011\rangle + \mid 100\rangle + \mid 101\rangle + \mid 110\rangle + \mid 111\rangle] \qquad (1)$$

If such a state is measured, it would collapse to one of the constituent states with a probability 1/8. It would not, however, be possible to infer from the result of such a measurement what the state of the system was before the measurement. For instance, if we obtained a state $\mid 000\rangle$ as a result of a measurement, it is as likely to have collapsed from the state of eqn. (1) as from a state $(1/\sqrt{8})(\mid 000\rangle + \sqrt{7} \mid 111\rangle)$.

An unitary operation which mixes the one qubit state is a Hadamard gate, which acting on a state $\mid 0\rangle$ gives a state $(1/\sqrt{2})(\mid 0\rangle + \mid 1\rangle)$ and acting on a state $\mid 1\rangle$ gives $(1/\sqrt{2})(\mid 0\rangle - \mid 1\rangle)$. This gate is useful in quantum computation because when each of the n bits of a state $\mid 0, 0, \ldots, 0\rangle$ is passed through a Hadamard gate, it produces an superposition of all possible n-qubit states. A two qubit state which is useful is known as a controlled-NOT gate (C-NOT) in which one of the qubits acts as the control while the other is the target. If the control bit is in state $\mid 0\rangle$ it leaves the target unaltered while when the control bit is $\mid 1\rangle$ it flips the target bit. Consider what would happen if a two qubit state $\mid 00\rangle$ is subjected to an operation depicted by the circuit of figure 2. Here the first qubit is passed through a Hadamard gate after which it acts as the control bit of a C-NOT gate which acts on the second bit as its target.

Note that when the control bit is 0 the target bit remains zero whereas when the control bit is 1, it becomes 1 . As a result the final two bit state becomes $(1/\sqrt{2})(\mid 00\rangle + \mid 11\rangle)$ which is an entangled state.

The natural basis for measuring two qubit states consist of four states $\mid 00\rangle, \mid 01\rangle, \mid 10\rangle, \mid 11\rangle$

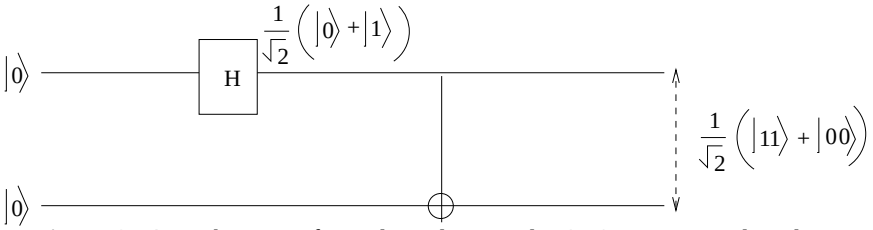

Figure 2. A combination of a Hadamard gate and a CNOT gate is used produce an entangled state.

which are known as computational basis. The states are un-entangled. However, by taking a suitable linear combination, one can define a different basis set. One such basis where the states of the two particles are entangled is known as the Bell basis. The basis states in this case are

$$\begin{aligned}
\phi^{+} &= \frac{|\,00\rangle + |\,11\rangle}{\sqrt{2}} \\
\phi^{-} &= \frac{|\,00\rangle - |\,11\rangle}{\sqrt{2}} \\
\psi^{+} &= \frac{|\,01\rangle + |\,10\rangle}{\sqrt{2}} \\
\psi^{-} &= \frac{|\,01\rangle - |\,10\rangle}{\sqrt{2}}
\end{aligned} \tag{2}$$

2.4. Problem of Measurement

The principle of superposition, which gives a quantum computer an immense advantage over a classical computer because of the former's ability to parallel process several states at a time, is also a cause of some concern. At the termination of an algorithm, one needs to measure the desired output. However, as the output is likely to be a component of a superposition, a measurement at the end of the process will only provide the correct output with a probability. It is possible to design algorithms such that the probability distribution at the end of the calculation is skewed in favour of the desired component. It will then be possible to run the algorithm repeatedly and get the correct output from the results occurring most often.

It was mentioned that the quantum states are coherent superposition of states. As long as such states are acted upon by unitary operators, the coherence will be maintained. To do so would require that the qubits should be isolated from stray particles of the environment. This, however, is not practically feasible as the qubits must be sufficiently open at least during the time of gate operations so that they can be controlled and manipulated externally. The decay and loss of phase coherence when the qubits interact with the environment is known as *decoherence*, which needs to be kept under control

There are a few problems for which quantum computers would offer a substantial speed up over classical computers. Peter Shor's algorithm [4] on prime factorization of composite numbers is one such problem. It may be remarked that a successful implementation of such an algorithm will have far reaching consequence on cryptographic protocol widely used for web transactions. The RSA public key distribution [11] relies on the fact that while it is simple to multiply to large prime numbers, the reverse problem, viz., factorization of a large composite number into its prime factors cannot be achieved in polynomial time. Thus, if one could devise an efficient polynomial time algorithm for factorization of a large composite number which is known to be a factor of two large prime numbers, it would effectively destroy the public key cryptographic protocol which is widely in use. A second algorithm which achieves a quadratic speeding over the corresponding classical algorithms is Grover's [8] search algorithm. In the following section we discuss this algorithm in some detail.

3. GROVER'S SEARCH ALGORITHM

Search algorithms are designed to locate an item having a defined characteristics from a database. A database may be structured or unstructured. For instance, in a telephone directory, the names are alphabetically arranged but the associated telephone numbers are randomly distributed. Such a database is structured with respect to names but unstructured with respect to the telephone numbers. Trying to find the name of a person from his telephone number from such a database is like searching for the proverbial "needle in the haystack !". One can formulate this problem mathematically as follows. Suppose we have a data base of $N = 2^n$ number of elements. We define a function$f(k)$ which is such that for all values of $k(0 \leq k \leq 2^n - 1)$ the function takes the value zero except that there exists a single k_0 for which $f(k_0) = 1$. If the database is random, one has to search through the entire database, evaluate the function $f(k)$for each value of k, until we find the value of k for which the function evaluates to unity. To locate with a probability of half we require $N/2$ number of trials, and the definitive search can require even $N - 1$ number of trials, for the extremely unlucky case when the last item of the database happens to be the one we are looking for. The number of trials is $O(N)$. A structured database will obviously reduce the effort.

Grover [8] designed an algorithm for searching for an element in a quantum computer, which speeds up the search quadratically, i.e. instead of the number of trials being $O(N)$, it requires only $O(\sqrt{N})$ number of trials. Since we are working with a quantum computer, we assume that we have $N = 2^n$ number of states, each having the same amplitude. For simplicity, we will take these states to form an orthonormal basis. With each item in our database we associate one basis state. One of these basis states is *marked*, i.e. it has certain properties we are looking for. The algorithm starts with an initial state $\mid s\rangle = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} \mid x\rangle$ which is a linear combination of the N states in which the amplitude of each of the basis states is the same, i.e., $1/\sqrt{N}$. The search is accomplished by selective amplification of the state corresponding to the item to be found. Let us call our marked state $\mid w\rangle$. To start with the amplitude of the marked state is also equal to $1/\sqrt{N}$.

Certain quantum operations have to be done on the standard state. These operations are executed

by quantum circuits corresponding to the unitary operators. The first of Grover's operations is to operate the standard state $\mid s\rangle$ with an operator which leaves all components of $\mid s\rangle$ other than that along $\mid w\rangle$ unchanged while the sign of the component along $\mid w\rangle$ is flipped. It is easy to visualize this in a vector diagram shown in Fig. 3. The figure has been drawn in two dimensions in the plane containing vectors $\mid s\rangle$ and $\mid w\rangle$ though the vectors $\mid s\rangle$ and $\mid w\rangle$ are in the N dimensional vector space. After the first rotation, the vector points in the direction indicated by $\mid s_1\rangle$. This is followed by a second rotation which takes $\mid s_1\rangle$ to $\mid s_2\rangle$ which has the component of $\mid s_1\rangle$ parallel to $\mid s\rangle$ unchanged but flips its sign perpendicular to it. It can be seen that the angle between$\mid s\rangle$ and $\mid s_2\rangle$ is 2θ *towards* the marked state $\mid w\rangle$.

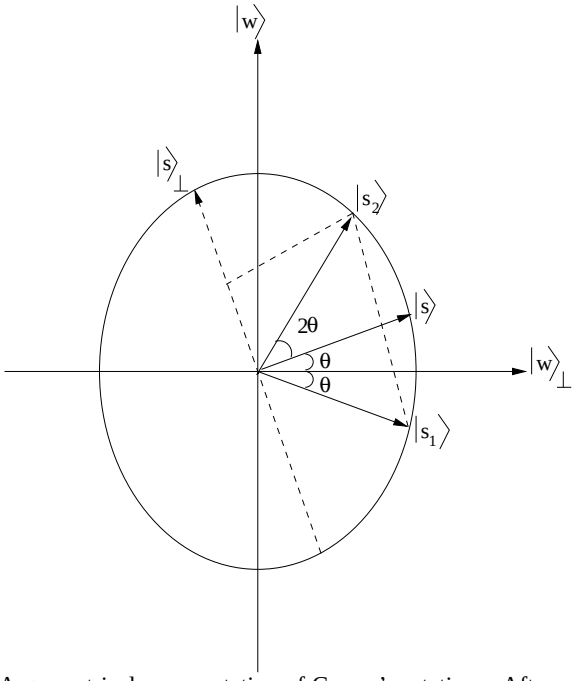

Figure 3. A geometrical representation of Grover's rotations. After one iteration the angle between the initial and the final directions of make an angle 2θ.

Consider the simplest case of N=4. Since the magnitude of each of the components in are equal, the cosine of the angle between $\mid w\rangle$ and $\mid s\rangle$ is $1/\sqrt{N}$. For $N = 4$, the angle is 60°. Thus after one Grover iteration, the marked state is found with certainty. For a general N, the same principle is valid. Recall that for large N, $\sin\theta \simeq \theta = 1/\sqrt{N}$. The number of iteration n required to align the vector $\mid s\rangle$ with the marked state $\mid w\rangle$ is given by

$$n \cdot 2\theta = \frac{\pi}{2}$$

which gives

$$n = \frac{\pi}{4\theta} \approx \frac{\pi}{4}\sqrt{N} \tag{3}$$

showing the quadratic speeding up of the search algorithm.

4. PRACTICAL RELIZATION OF A QUANTUM COMPUTER

Practical realization of a quantum computer is a challenging problem. Simply stated, a quantum computer is a device which uses quantum states to encode, process and store information. In principle ,any two level system, such as, a spin half particle or a two level atomic system can be used for preparing a qubit. However, in practice, a set of criteria known as DiVincenzo [12] criteria are considered to be crucial for building a physical quantum computer. These criteria are summarized in the following.

1. A scalable physical system with well characterized qubits : We have seen that qubits are essentially the workhorse of a quantum computer. We need them for encoding and storing of information. As we need to manipulate the qubits externally, it is necessary to clearly identify the states which define qubits and their interaction with other states of the system, with states of other qubits and with external fields. For instance, if we take a qubit to be defined as the two lowest lying states of the system, the probability of transition to higher lying states should be negligible. A wide range of physical systems can be used as qubits. These include (i) nuclear spins addressed through nuclear magnetic resonance , (ii) hyperfine or Zeeman sub-levels in electronic ground states (iii) single photon with polarization states (vertical and horizontal) representing the up and down states, (iv) electron spins (v) flux qubits in superconducting Josephson junctions etc.

2. Ability to initialize the state of the qubits to a simple fiducial state such as : This is identical to the requirement of initializing the registers in a classical computer. Registers would otherwise contain “garbage” and the result of computation become untrustworthy. In many applications such initializations can be simply done by cooling the atoms to a low temperature ensuring that the atoms remain in their ground state. This may not, however, be always possible (e.g. in liquid state NMR) in which case one can use a thermally populated state as the initial state.

3. Long decoherence times, much longer than the gate operation time : Earlier in the article we have commented on loss of coherence due to interaction of the qubits with environment. Decoherence is important as it is the time required for a quantum system to interact with environment and go over to a classical regime. After this. the system cannot evolve as a quantum system. As the qubits must be externally manipulated for logic gate operations, they have to be exposed to environment during such process. The decoherence time should, therefore, be long compared to gate operation time. In many applications the decoherence times may be of the order of microseconds. As gates can be implemented in as short a time as a pico-second, we can still perform about a million operation on the system before decoherence sets in. There has to be some tradeoff between the need to expose the qubit to the environment and decoherence that must inevitably set in. Such non-controllable errors are even known in classical computing. The idea of a fault tolerant computing is to retrieve right information from a noisy channel through error correcting codes. Current classical computers can tolerate 0.01% error in data. Quantum computers still have a long way to go in this direction.

4. A “universal” set of quantum gates : Data manipulation in a computer (classical or quantum) is done using logic gates. In quantum computers such gates would mimic unitary operators, which acting on a quantum state would give rise to a new state. As mentioned earlier, unlike classical gates, the quantum gates are reversible. This implies that the number of input qubits is equal to the number

of output qubits. Several one-qubit gates have been designed. Hadamard gate, which acting on a state $\mid 0\rangle$ gives a symmetric combination of the states $\mid 0\rangle$ and $\mid 1\rangle$,and, which acting on a state $\mid 1\rangle$ gives the anti- symmetric combination of the same is widely used in quantum circuits. A sequence of two Hadamard gates can be used to simulate a beam splitter depicted in Fig. 1. Yet another one-qubit gate is the NOT gate. CNOT gate, described earlier is an example of a two-qubit gate. There are gates with three input-output qubits as well. It is well-known that in classical circuits, the NAND gate is universal, i.e., any logic operation can be performed using NAND gates alone. Several universal family of quantum gates have been identified. One such family consists of the Hadamard gate, the CNOT gate, the phase gate and the $\pi/8$ gate, the last mentioned gate generates a phase difference of $\pi/4$ between the bits 0 and 1.

5. A qubit specific measurement capability : We have briefly touched upon the problems related to measurement process. The measurement process depends on the system under consideration. In most cases projective measurements are the most commonly used method to extract the output. In some cases (e.g. in liquid state NMR computer) such measurements are not feasible and one makes ensemble average measurements.

There are two more subsidiary criteria which must be satisfied if the quantum computers are to be networked. We will not go into these additional criteria.

At present experimental realization of a practical quantum computing device are based on the following techniques :

(i) NMR in both liquid and solid states

(ii) Coupled atoms and photon in an optical cavity, the so-called cavity-QED

(iii) Trapped neutral atoms or ions

(iv) Semiconducting quantum dots

(v) Superconducting Josephson junctions

(vi) Photonic circuits.

Shor's factorization algorithm has been successfully implemented on a seven qubit NMR machine while the same has been implemented with five qubits on a photonic chip. While a full-fledged quantum computer still looks like a distant dream, development in the field of computer science has been known to leapfrog. A quantum computer, therefore, may become a reality in not so distant a future.

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P R A Y A S
Students' Journal
of Physics

Effects of Gravity on Spin Zero particles

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Communicated by: A.M. Srivastava

Relativistic theory of quantum fields is primarly written in a flat background, i.e. in absence of gravity. This raises the obvious question that what would happen to this theory if the background spacetime is curved. The answer to this question comes most easily from the principle of general covariance which is a very efficient way to introduce the effects of gravitation into any theory of Physics that can be written in a covariant language. In this work we shall apply the principle of general covariance on Klein-Gordan equation and see what are the principle changes that come into picture.

1. PRINCIPLE OF EQUIVALENCE

The principle of Equivalence rests on the equivalence between the inertial mass and gravitational mass. In Principia, Sir Issac Newton distinguished between these two kind of mass parameters by their appearence in the respective laws. The mass parameter that appears in the Newton's second law is known as the inertial mass and the mass parameter that appears in the force law of gravitation is known as the gravitational mass. There is no reason to believe in any correspondence between these two mass parameters. However precise experiments suggest that the ratio of these two parameters does not differ from particle to particle by more than one part in a billion. Einstein took this equivalence very strongly and this lead him to establish an equivalence between gravity and acceleration. This is the key idea behind the "principle of equivalence". Formally, this principle states that[1]

> At every space-time point x^μ in an arbitrary gravitational field it is possible to choose a locally inertial coordinate system ($\xi^\mu(x^\nu)$) such that, within a sufficiently small region of the point in question, the laws of nature take the same form as in unaccelerated Cartesian coordinate systems in the absence of gravitation.

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Therefore in this small neighbourhood, we can write down the usual 'flat spacetime laws of physics' in terms of the coordinate $\xi^\mu(x^\nu)$ and then invert the coordinates to get the 'generalised laws of physics' in terms of the coordinates x^μ. By doing so the effects of gravitation are incorporated into the laws.

In a flat background the invariant distance in spacetime is measured *via* the metric tensor $\eta_{\mu\nu}$ through the equation:

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu. \tag{1}$$

For generalised coordinates we use the metric tensor $g_{\mu\nu}$ instead of $\eta_{\mu\nu}$. This means that operations such as contraction and raising and lowering of indices, will now require the use of metric tensor $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$. A second point that we notice here is that derivatives of vectors (and higher rank tensor fields) do not transform as tensors under general coordinate transformation. Therefore we introduce the notion of covariant derivatives which retains the tensor character of the equation and pave a clean way of introducing gravity into the problem. Covariant derivative(denoted by a semi-colon) of a contravariant tensor is given by:

$$A^\mu{}_{;\nu} = A^\mu{}_{,\nu} + \Gamma^\mu_{\sigma\nu} A^\sigma, \tag{2}$$

and that of a covariant tensor is given by:

$$A_{\mu;\nu} = A_{\mu,\nu} - \Gamma^\sigma_{\mu\nu} A_\sigma, \tag{3}$$

where the comma denotes ordinary derivatives and the Γ's are the affine connections that encode all the information about gravity through the metric tensor $g_{\mu\nu}$:

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}(g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho}). \tag{4}$$

This modification changes the calculus of vector fields, i.e. we need to generalize the concept of divergence, curl, Laplacian, etc. It is to be noted that the derivatives of scalar functions or gradients are ordinary vectors and therefore retain their form under the introduction of gravity, i.e. given a scalar function $\Phi(x^\mu)$,

$$\Phi_{;\mu} \equiv \Phi_{,\mu}. \tag{5}$$

Covariant curl also remains unchanged because of the fact that $\Gamma^\sigma_{\mu\nu}$ is symmetric in μ and ν:

$$A_{\mu;\nu} - A_{\nu;\mu} = A_{\mu,\nu} - A_{\nu,\mu}. \tag{6}$$

Divergence of a vector field, however, changes in a non-trivial manner. The divergence of a vector field A^μ is $A^\mu_{;\mu}$. From Eq.(2) we have

$$A^\mu_{;\mu} = A^\mu_{,\mu} + \Gamma^\mu_{\mu\sigma} A^\sigma. \tag{7}$$

We can simplify $\Gamma^{\mu}_{\mu\sigma}$ using Eq.(4) to get:

$$\Gamma^{\mu}_{\mu\sigma} = \frac{1}{2} g^{\mu\rho} g_{\rho\mu;\sigma} \tag{8}$$

If we now treat the metric tensor as a matrix then is equation can be further simplified using the following identity from matrix algebra:

$$\text{Tr}\left[M^{-1}(x)\frac{\partial}{\partial x^{\lambda}} M(x)\right] = \frac{\partial}{\partial x^{\lambda}} \ln \text{Det}\, M(x), \tag{9}$$

where $M(x)$ is an invertible square matrix with entries as functions of x^{λ}. Using this identity Eq.(8) becomes:

$$\Gamma^{\mu}_{\mu\sigma} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^{\sigma}} \sqrt{g}, \tag{10}$$

where g is the modulus of the metric tensor $g_{\mu\nu}$. Plugging this into Eq.(7) we finally arrive at a covariant form of the divergence:

$$A^{\mu}_{;\mu} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^{\sigma}} \left(\sqrt{g} A^{\sigma}\right). \tag{11}$$

The generalised Laplacian can be modified using the divergence. Laplacian of a scalar field Φ is given by $\Phi^{,\mu}_{\ ,\mu}$. When we covariantize this form it becomes $\Phi^{;\mu}_{\ ;\mu}$. From Eq.(5) this simplifies to $\Phi^{,\mu}_{\ ;\mu}$. The first partial derivative of Φ in this expression is an ordinary vector. Therefore we can make use of Eq.(11) to simplify this expression further to get the covariant form of the Laplacian as:

$$\Phi^{;\mu}_{\ ;\mu} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^{\mu}} \left(\sqrt{g} g^{\mu\nu} \Phi_{,\nu}\right). \tag{12}$$

We are now in position to correctly write down the Law of Physics in the presence of gravity. Take the equation representing the Law and perform the following changes to it:

- Replace all $\eta_{\mu\nu}$ by $g_{\mu\nu}$.
- Replace all ordinary derivatives by their covariant forms.

This recipe for introducing gravitation into the problem (which resulted from the equivalence principle) is alternatively known as "Principle of General Covariance". Formally, it states that a Law of Physics holds in a general gravitational field if the following conditions are satisfied:

- The Law holds in the absence of gravitation.
- The Law has form invariance under general coordinate transformation (i.e. the law is expressible as a tensor equation.)

Having obtained this recepie we are now in a position to apply the principle of general covariance to any physical theory written in flat spacetime. This principle finds application in mechanics and electrodynamics[1]. Here we apply it to Klein Gordon equation.

2. KLEIN GORDON EQUATION IN THE PRESENCE OF GRAVITATIONAL FIELD

A massive relativistic spin zero particle is adequately described by the Klien-Gordan equation which reads:

$$(\partial_\mu \partial^\mu - m^2)\Phi = 0, \tag{13}$$

where m is the mass of the field quanta in case the theory is quantized.Here we have used the signature of the metric as $-+++$ (if we use the reciprocal signature then sign of the mass term will get simply reversed). A direct application of the principle of general covariance modifies this equation to its covariant form as follows:

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left(\sqrt{-g} g^{\mu\nu} \Phi_{,\nu} \right) - m^2 \Phi = 0, \tag{14}$$

where the negative sign preceeding g is to ensure that the quantity inside the square root is positive. This is the general wave equation for a massive scalar field "propagating freely" in a curved background. We can simplify this equation for different backgrounds. At this stage we have not assumed anything about the strength of the gravitational field. The only assumption made here is that field is specified by some external matter distribution and remains unaffected by the motion of the Klein Gordon particle itself.

2.1. Klein Gordon equation from Action Principle

Action is a scalar quantity that has units of 'Joules second'. Curved spacetime and flat spacetime are related to each other by invertible coodinate transformations which form a subset of general coordinate transformation. In the ensuing discussions we investigate the invariance of action under general coordinate transformations.

In flat space (i.e. in the absence of gravity) action is simply the 4-volume integral of the lagrangian density.

$$\mathcal{S} = \int \mathcal{L} \mathrm{d}^4\mathrm{x} \tag{15}$$

The Lagrangian density $\mathcal{L}$ is a locally defined scalar field. Therefore it should remain invariant under general coordinate transformation. The 4-volume element is however not a scalar under general coordinate transformation. It is a scalar density of weight 1. A scalar density s is a quantity which transforms as

$$s \rightarrow s' = |\mathcal{J}|^{\mathrm{w}} \mathrm{s}, \tag{16}$$

where $\mathcal{J}$ is the Jacobian of the transformation and w is the weight of the scalar density. This therefore destroyes the scalar property of the action. To remedy this we notice that the correct volume element is now given by $|\mathcal{J}|\mathrm{d}^4\mathrm{x}$. Consequently, the action changes to

$$\mathcal{S} = \int \mathcal{L}|\mathcal{J}|\mathrm{d}^4\mathrm{x} \tag{17}$$

We can write this measure in terms of the determinant of the metric tensor. This is a more convenient form particularly for minimising the action at a later stage. We notice that

$$d^4x' = |\mathcal{J}|\mathrm{d}^4\mathrm{x}, \tag{18}$$

and

$$g' = |\mathcal{J}^{-\infty}|^2\ \mathrm{g}, \tag{19}$$

where g is modulus of the determinant of the metric tensor $g_{\mu\nu}$. This shows that that product $\sqrt{g}\ d^4x$ is the correct invariant integral measure. Therefore the correct action in a curved space is

$$\mathcal{S} = \int \mathcal{L}\sqrt{\mathrm{g}}\ \mathrm{d}^4\mathrm{x}. \tag{20}$$

All that remains now is to write down the correct form of the Lagrange density $\mathcal{L}$. To do this from scratch we revert to the mathematical statement of the "equivalence principle" namely the "principle of General Covariance". Take the Lagrangian $\mathcal{L}$ and perform the following changes to it:

- Replace all $\eta_{\mu\nu}$ by $g_{\mu\nu}$.
- Replace all ordinary derivatives by covariant derivatives.
- If a tensor density appears in the Lagrangian, append the appropriate weight factor.

Having obtained this recepie we can now write the general action (20) for various physical systems and investigate the effects of gravitation on it. Our concern is the Klein-Gordan field. We notice that it can be derived from minimizing the following action:

$$\mathcal{S} = \int \mathrm{d}^4\mathrm{x}\mathcal{L} = \int \left\{\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\mathrm{m}^2\Phi^2\right\}\mathrm{d}^4\mathrm{x}. \tag{21}$$

In a curved spacetime this action is modified in accordance to the principle of general covariance and we have

$$\mathcal{S} = \int \sqrt{-\mathrm{g}}\mathrm{d}^4\mathrm{x}\mathcal{L} = \int \sqrt{-\mathrm{g}}\left\{\frac{1}{2}\mathrm{g}_{\mu\nu}\partial^\mu\Phi\partial^\nu\Phi - \frac{1}{2}\mathrm{m}^2\Phi^2\right\}\mathrm{d}^4\mathrm{x}. \tag{22}$$

Considering the background to be static(i.e. independent of the motion of the KG particle itself), we vary this action with respect to the wavefunction Ψ to get the variation in the action:

$$\delta\mathcal{S} = \int \sqrt{-\mathrm{g}}\left\{\mathrm{g}_{\mu\nu}\partial^\mu\Phi\partial^\nu\delta\Phi - \mathrm{m}^2\Phi\delta\Phi\right\}\mathrm{d}^4\mathrm{x}. \tag{23}$$

Integrating the first term by parts and throwing away the boundary term we get:

$$\delta\mathcal{S} = \int \left(\partial^\nu(\sqrt{-\mathrm{g}}\mathrm{g}_{\mu\nu}\partial^\mu\Phi) - \sqrt{-\mathrm{g}}\mathrm{m}^2\Phi\right)\delta\Phi\ \mathrm{d}^4\mathrm{x}. \tag{24}$$

This variation in the action should be zero for any varaition $\delta\Phi$. Therefore we set the integrand to zero and we get back equation 14.

3. SIMPLIFICATION OF THE WAVE EQUOATION IN A WEAK FIELD APPROXIMATION

In the weak field approximation we introduce first order pertubations over the flat metric, i.e.

$$g^{\mu\nu} \simeq \eta^{\mu\nu} + h^{\mu\nu} \tag{25}$$

where $|h^{\mu\nu}| \ll 1$. Therefore in the calculations that follow we ignore terms that are second and higher order in $h_{\mu\nu}$.

The determinant g of the metric tensor is

$$g = -(1+h)$$
$$\text{where } h = \eta^{\mu\nu} h_{\mu\nu}.$$

On using this approximation to simplify Eq.14 we get

$$\partial^\mu \partial_\mu \Phi - m^2 \Phi + \left\{ \frac{1}{2} \partial_\mu h \, \partial^\mu \Phi + \partial_\mu h^{\mu\nu} \, \partial_\nu \Phi + h^{\mu\nu} \, \partial_\mu \partial_\nu \Phi \right\} = 0.$$

The terms inside the curly bracket is the correction arising from the effects of gravity. It is to be noted here that the raising and lowering of indices is now to be done with respect to $\eta_{\mu\nu}$ instead of $g_{\mu\nu}$. This is the scalar wave equation in linearised gravity. We will consider further simplification of this equation in a particularly simple form of the metric tensor.

3.1. Wave equation in nearly newtonian spacetime

If the sources are weak then we can approximate the metric tensor (to the first order in ϕ) as

$$g_{\mu\nu} = \text{diag}(-1-2\phi, 1-2\phi, 1-2\phi, 1-2\phi), \tag{26}$$

where the monopole contribution to ϕ is

$$\phi = -\frac{M}{r}. \tag{27}$$

Consequently, the pertubation over the flat metric is

$$h_{\mu\nu} = -h^{\mu\nu} = \text{diag}(-2\phi, -2\phi, -2\phi, -2\phi) \tag{28}$$

and

$$h = \eta^{\mu\nu} h_{\mu\nu} = -4\phi. \tag{29}$$

If we use these as inputs, then Eq(28) reduces to:

$$-(1+2\phi)\partial_t^2 \Phi + (1-2\phi)\nabla^2 \Phi - m^2 \Phi = 0. \tag{30}$$

This equation is appropriately weighted by factors that appear in the metric tensor. We notice that this equation reduces to the Eq (13) in the limit $\phi \to 0$. We do not attempt to solve this equation here but only mention that it exhibits a solution that is spherically symmetric and the radial part is a Bessel's type differential equation. The closed form of this differential equation is not known however series solution of a more general equation[1](although complicated) are available[2].

4. SUMMARY

We emphasize over here that the treatment done in this work is semi-classical in nature. It is appropriate for a senario where the general setting is that we treat matter particles quantum mechanically and gravitational field classically. Such a theory is a preliminary attempt towards a more complete theory of quantum gravity. Similar attempts have already been done earlier where electromagnetic field is considered as a classical field interacting with quantized matter. Such semi-classical calculations yeilds results that are in agreement with the complete theory of quantum electrodynamics. This gives us some hope that even though we do not have a full fleged theory of quantum gravity, we certainly can predict few aspects, if not all, of the influence of gravitational field on quantum phenomenon. In this paper we dealt with spin zero particles. However this formalism can be extended to theories of higher spins.

ACKNOWLEDGEMENT

I would like to thank Prof. S.C. Phatak for encouraging me to work out this problem. This problem grew out of an assignment in our special relativity class. I had really exciting discussions with Prof. Phatak. We thought over many different ways to arrive at a precise bound state solution to the wave equation in a Schrodinger background. This work promises a lot of scope on the Physics of quantum particles bounded gravitationally to a black hole. I would also like to thank Dr. Yogesh Srivastava who gave me valuable inputs towards the end of the project.

References

[1]If we simplify the general wave equation in a Schwarzschild background then in a weak field expansion this equation reduces to Eq.(33). The radial equation is

$$\frac{d^2R(r)}{dr^2} + \left(\frac{1}{r-r_s} + \frac{1}{r}\right)\frac{dR(r)}{dr} + \left(\frac{E^2r^2}{(r-r_s)^2} - \frac{m^2r}{r-r_s} - \frac{l(l+1)}{r(r-r_s)}\right)R(r) = 0, \qquad (31)$$

where $r_s = 2M$. Although closed form solutions to the radial equation are not yet known in Mathematical Physics but there have been attempts to find a series solution near the critical points[2].

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Characterization of instabilities in discharge plasma

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Communicated by: A.M. Srivastava

Characterization of deterministically chaotic physical systems are of greater importance. Dynamics of a system will reflect on the time dependence of certain easily measurable quantities. The temporal development of such quantities is known as the time series. Time series analyses can give greater insight into the dynamics of the system. In this paper we report the characterization of instabilities in discharge plasma by evaluating Lyapunov exponents from time series obtained.

1. INTRODUCTION

Nonlinear dynamics and chaos theory started with the intention of investigating the qualitative behavior of nonlinear problems which were difficult to solve analytically.

Phenomena which have no clear relation between cause and effect are said to possess random element. Randomness is fundamental that gathering more information does not reduce randomness. Randomness gathered in this way has come to be called chaos[1]. A chaotic system has a very sensitive dependence on initial conditions. Chaos emerges from the theory of dynamical systems.

2. INSTABILIES IN DISCHARGE PLASMA

Discharge plasma is a typical nonlinear dynamical system with a large number of degrees of freedom [2]. It is an interesting medium to test the universal characteristics of chaos. Non linear analysis of gaseous discharge and plasma derive from their potential applications in the development of laser devices, controlled fusion etc. where the problems of instabilities and turbulence are very important. The study of chaotic behavior in gas discharge also enables one to understand the reproducibility of the plasma conditions in laboratory plasma experiments, and their sensitive dependence on initial

conditions. Even today a quantitative or an accepted qualitative explanation of the nonlinear behavior of dc discharge plasma has not been given. Though an exact description of the mechanism responsible for the appearance of oscillatory behavior is not available, it is expected that the chaotic behavior is generated from macroscopic properties of the discharge. These oscillatory behavior of the discharge is a kind of self generated oscillations because the plasma system is not driven by any external periodic forces. The fundamental frequency of self oscillations varies with the change in the control parameters like discharge current.

The dynamics of a system will reflect on the time dependence of certain easily measurable quantities. The temporal development of such quantities is known as time series.Time series analysis reveals the characteristics of instabilities. A time series is a sequence of data points of an observed variable at equally spaced time intervals and time series analysis comprises of methods that attempt to understand such time series. Analyses enable one to understand the underlying context of the data points like where did they came from and what generated them or to make predictions. In the present study time series analysis is employed for the characterization of instabilities.

Cheung et.al.[1] described a qualitative representation of oscillatory phenomenon in dc discharge. When an anode is biased positively with respect to the cathode, energetic electrons are ejected from the cathode. The electrons periodically ionize the background neutral gas and create plasma between the electrodes. The generation of primary electrons from the cathode and the production of plasma are strongly coupled. The primary electrons ionize the gas and sustain the plasma while the plasma reduces to negative space charge and facilitates electron emission. By varying the plasma discharge parameters one can control this coupling or the feedback process and the resulting plasma dynamics can be made unstable. This occurs when the plasma potential is negative with respect to the anode where the potential is unstable and current oscillations occur.

The rate of plasma formation (determined by the rate of neutral ionizations by primary electrons and the plasma decay time) can be written as

$$\frac{dn_0}{dt} = n_\rho N_n < \sigma V_\rho > -\frac{n_0}{\tau} \tag{1}$$

where n_o and N_n are the density of the plasma electrons and the neutral atoms respectively, n_ρ and V_ρ are the density and velocity of the primary electrons, σ is the ionization cross-section and τ is the plasma decay time.Once a discharge is initiated the primary electron flux $J_\rho = n_\rho V_\rho$ increases rapidly and the entire voltage is confined in a narrow potential sheath that exists between the plasma and the electrodes. The width of the sheath structure is typically of the order of tens of Debye length λ_D. In the steady state primary electron flux $J_\rho \propto \lambda_D^{-2}$ and the efficiency of primary electrons depends on how fast plasma ions can drift to form a potential sheath. An approximate rate equation for the primary electron emission

$$\frac{dn_0}{dt} = \sigma n_\rho (u_d/L') \tag{2}$$

where σ is constant,u_d is the ion drift speed and L' is the effective plasma radius. The above equation along with the angular discharge repetition frequency were well studied and have been

shown to display chaotic behavior [2]. To maintain a stable sheath, the plasma ions have to enter the sheath from the plasma side with a minimum of drift speed $u_d \geq c_s$, the ion acoustic speed. The ratio of the ion flux to primary electron flux is $J_i/J_\rho = (m_e/m_i)^{1/2}$ where m_e/m_i is the electron to ion mass ratio. The maximum ion flux generated through ionization is approximately given by

$$[\frac{J_i}{J_\rho}]_{ion} = N_n \sigma'_L = \frac{L'}{I_m} \tag{3}$$

where I_m is the mean free path. This ion flux must be large enough to neutralize the negative space charge due to primary electrons and sustained sheath. As a result if $L'/I_m \geq (m_e/m_i)^{1/2}$ both the discharge current and the sheath are destabilized. The destabilizing process develops through the accumulation of negative space charge and the depression of the plasma potential to negative values forming a virtual cathode in the plasma. As a result, the effective energy of a primary electron is no longer a constant, but depends on the spatial and temporal evolution of the plasma potential. This in turn affects the mean free path l_m and the particle flux J_i/J_ρ .In this unstable state inherent shot to shot noise fluctuations of $\delta_n/n \leq 0.1\%$,which make only a negligible change in the initial discharge condition, cause a considerable change in the plasma and lead to chaotic behavior.

3. CHARACTERIZATION OF INSTABILITIES

Time series analyses can give greater insight into the dynamics of the system. Similar analyses have been carried out in different systems involving nonlinearities. We describe one of the methods used in time series analysis so as to study the dynamics of a non linear system, determination of Lyapanov exponent.

One of the interesting nonlinear systems in the context of experimental investigations is gaseous plasma. Discharge plasma possess a large number of degrees of freedom and is an interesting medium to test some of the universal characteristics of chaos. For example, in the case of discharge plasma one can monitor the discharge current to get a time series.

A discharge cell has been designed for the present study. In our observations, as we have changed the discharge current for which different series of discharge instabilities were seen, with different frequencies.

4. EXPERIMENTAL SET-UP

The schematic of the experimental set-up is given in figure 1. The cell consists of a glass tube of 1 cm diameter socketed into two metal caps made of stainless steel. Separation between the ends of the caps is 3 cm and they act as electrodes. One of the cylindrical cap is provided with a glass window with O ring for effective sealing. The tube is provided with gas inlet and outlet ports. Desired gas can be fed through a needle valve and the cell is operated as a continuous flow discharge cell by connecting the outlet to a diffusion vacuum pump.

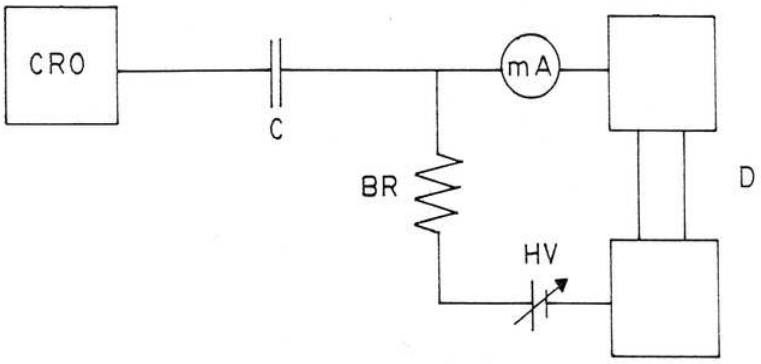

Figure 1. Schematic of the experimental setup D− discharge cell, HV− high voltage power supply, BR− high resistance, C− coupling capacitor, CRO− Digital Storage Oscilloscope

Discharge was generated using a low noise high voltage power supply (Stanford $PS325$). By optimizing discharge current and pressure in the discharge cell instabilities were generated. The instabilities developed across the load resistor was fed to the Digital Storage Oscilloscope through a coupling capicitor(0.1μ F). The capacitor blocks the dc voltage and ac signal is directly fed to the oscilloscope.

To extract the relevant time series from the discharge, current was monitored using the digital storage oscilloscope (Aplab $D36000A4$ series) interfaced to a computer through its RS 232 port. Data were stored in the oscilloscope and digitized data was directly fed to the computer and saved. The digitization of the data was carried out at suitable time interval. In this study the time series of the pattern shown in fig.2 is considered.

5. LYAPUNOV EXPONENT – SIGNATURE OF CHAOS

For the nonlinear time series analysis it is of great interest to measure the Lyapunov characteristic exponents which, if positive, are the most striking evidence for chaos. Many people had devised different techniques and algorithms for the computation of Lyapunov exponents.

Lyapunov exponent of a given trajectory characterize the mean exponential rate of divergence of trajectories surrounding it. It is a measure of sensitivity to initial conditions. A positive Lyapunov exponent may be taken as a definition of chaos.

Here we take an algorithm to calculate maximal Lyapunov exponent proposed by HolgerKantz [3]. The basic idea of this method is that the distance between the two trajectories typically increases with a rate given by the maximal Lyapunov exponent. One looks for a point of the time series which is closest to its first point. This is considered as the beginning of a neighboring trajectory, given by the consecutive delay vectors. Then computing the distance between these two trajectories in time. When the distance exceeds some threshold, for this point of the time series a new trajectory

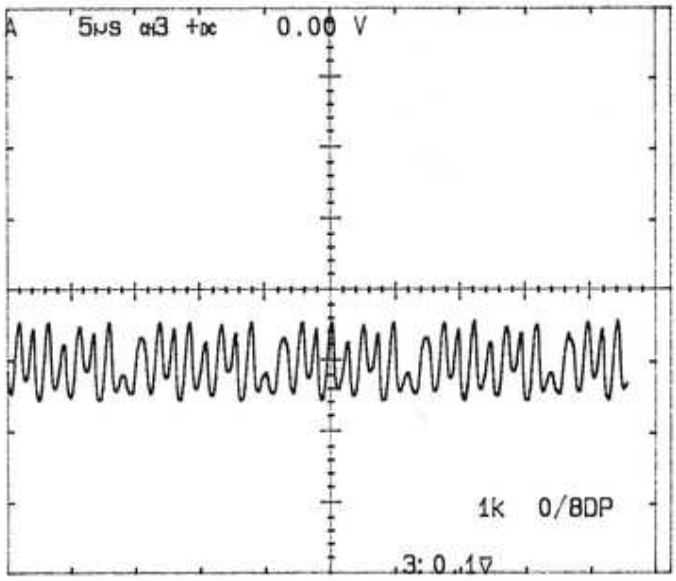

Figure 2. instability pattern obtained

is searched for, when distance is as small as possible under the constraint that the new difference vector points more or less into the same direction as the old one. The logarithms of the stretching factors of the difference vectors are averaged in time to yield the maximal Lyapunov exponent.

In order to measure the maximal Lyapunov exponent we fix t, and search for all neighbors x_i inside an ϵ neighborhood U_t and compute the average of distance between all neighboring trajectories and the reference trajectory x_t as a function of τ . τ is the relative time referring to the time index of the starting point. To get rid of the fluctuations we take the logarithm of these average distances, which yields the local effective Lyapunov exponent plus a fluctuation given by the angle ϕ . Now this can be averaged in t over the full length of the time series . The local angles are averaged out and the effective exponents are averaged to the true can be done very fast and is given by

$$S(t) = \frac{1}{T}\sum_{t=1}^{T} ln(\frac{1}{|U_t|}) \sum_{i \in U_t} dist(x, x, \tau) \tag{4}$$

Initially the difference vectors in the phase space are pointing in any direction, therefore the distance behaves like

$$dist = \sum_i a_i exp(\lambda_i t) \tag{5}$$

where λ_i are the effective Lyapunov exponents in the stable and unstable directions. For an intermediate range of τ , $S(\tau)$ increases linearly with the slope λ which is the estimate of the maximal

Lyapunov exponent. This is the scaling range, where on the one hand τ is large enough such that nearly all distance vectors point into the unstable direction and on the other hand the corresponding distances $dist(\tau)$ are smaller than the size of the attractor. When they approach the size of the attractor, $S(\tau)$ asymptotically tends towards a constant, since the distance cannot grow more.

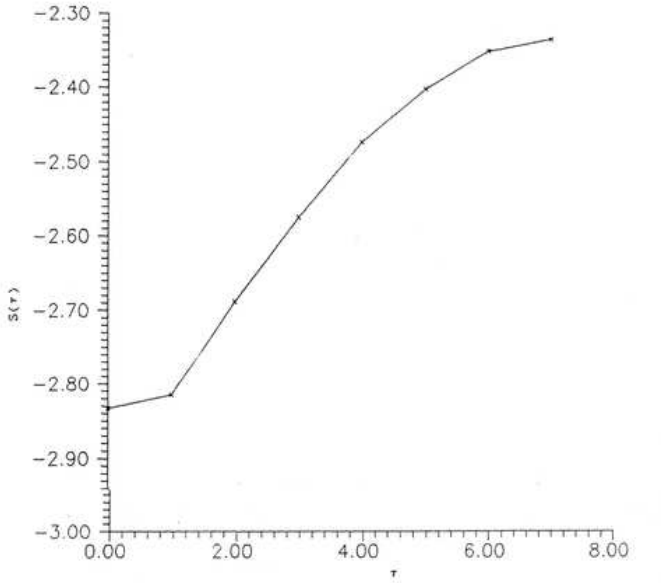

Figure 3. plot of $S(\tau)$ versus for the time series in figure

If the data are noisy, the typical distance between two nearby trajectories is of the order of the noise level. If we choose ϵ smaller than the noise amplitude and if we find neighbors for this value, $S(\tau)$ jumps from a value smaller than $ln\epsilon$ to a value given by the noise level at $\tau = 1$. If this value is not too large, one can still find a scaling range and the exponents thus found is not affected by the noise.

The numerical value for the maximal Lyapunov exponent is the slope of the curve $S(\tau)$ in the scaling region. Lyapunov exponent was calculated for some of the selected data. A typical plot of $S(\tau)$ versus τ is given in $fig.3$ .The slope of which is (0.09 ± 0.02). This small value of the Lyapunov exponent shows that the dynamics is not in the chaotic regime, but only onset of chaos.

6. CONCLUSION

A discharge cell for instabilities study in discharge plasma has been designed and fabricated. Making use of the discharge cell an experimental system has been optimized for the study. It has been found that at an optimized pressure the discharge current decides the onset of randomness in the discharge. Random signals generated were recorded using the digital storage oscilloscope. Lyapunov exponent is a signature of chaos. In the present study Lyapunov exponent is found to be a positive quantity,

indicating that there is an onset of chaos. One of the signals generated was subjected to time series analysis.

Acknowledgement

The authors are thankful to Kerala State Council for Science,Technology and Environment,Govt. of Kerala for financial assistance in the form of a student project.

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PROBLEMS IN PHYSICS

Readers are invited to submit the solutions of the problems in this section within two months. Correct solutions, along with the names of the senders, will be published in the alternate issues. Solutions should be sent to: H.S. Mani, c/o A.M. Srivastava, Institute of Physics, Bhubaneswar, 751005; e-mail: ajit@iopb.res.in

Communicated by H.S. Mani

1. We know that the parity operator $\hat{P}$, is defined on the eigenstates $|x>$ of the position operator $\hat{x}$, as

$$\hat{P}|x> = \ |-x>$$

.

From this we can show the eigenstates $|p>$ of the momentum operator $\hat{p}$ transform as

$$\hat{P}|p> = \ |-p>$$

. Construct the parity operator in terms of $\hat{x}, \hat{p}$, such that

$$\hat{P}\hat{x}\hat{P^{-1}} \ = \ -\hat{x}$$

and

$$\hat{P}\hat{p}\hat{P^{-1}} \ = \ -\hat{p}$$

Treat the problem, for simplicity, as one-dimensional.

(You need to introduce a constant of dimensions length, however the result will be independent of the choice you make)

2. If an operator $\hat{A}$ commutes with $\vec{J}.\hat{n}$ and $\vec{J}.\hat{m}$, where $\vec{J}$ is the angular momentum operator and $\hat{n}$ and $\hat{m}$ are two linearly independent vectors, show that $\hat{A}$ commutes with all three components of angular momentum,

$$[\hat{A}, \vec{J}] \ = \ 0$$

Solutions to the problems given in Vol. 4 No. 3

Problem 1: Consider a hydrogen atom confined inside a thin uncharged conducting shell of radius R. Assume $R >> a_H$, where a_H is the Bohr radius. The proton (assumed infitely heavy) is at the centre of the shell.

Find the first nonvanishing correction to

(a) The radius of the hyderogen atom assuming Bohr quantization rule.

(b) The energy of the ground state.

Solution to Problem 1:

We use the method of images to solve the problem. If the elecron (charge $-e$) is at r from the proton, we have an image charge at R^2/r of value eR/r Thus we have, for the elctron,

$$\frac{mv^2}{r} = \frac{e^2}{4\pi\epsilon_0 r^2} - \frac{e^2 R/r}{4\pi\epsilon_0[R^2/r - r]^2}$$

$$= \frac{e^2}{4\pi\epsilon_0}[\frac{1}{r^2} - \frac{Rr}{[R^2 - r^2]^2}]$$

We also have

$$mvr = \hbar$$

Here m, v refer to the electron's mass and speed respectively.$\hbar$ is the Planck's constant devided by 2π as usual. Eliminating v from the two eqations and simplifying we get

$$r = \frac{a_H}{[1 - \frac{r^3}{R^3(1-(\frac{a_H}{R})^2)^3}]}$$

where

$$a_H = \frac{\hbar^2 4\pi\epsilon_0}{me^2}$$

To the lowest nonvanishing order we get

$$r = a_H(1 + (\frac{a_H}{R})^3)$$

The enrgy of the system is

$$E = \frac{mv^2}{2} - \frac{e^2}{4\pi\epsilon}(\frac{1}{r} - \frac{R/r}{R^2/r} + \frac{R/r}{R^2/r - r})$$

which can be simplified in a straight forward way leading to (to the lowest nonvanishing order)

$$E = -\frac{e^2}{4\pi\epsilon_0}\frac{1}{2a_H}(1 + 2(\frac{a_H}{R})^3)$$

Problem 2: A square cardboard of length L is intially at $x = 0$ with its corners at $(0, 0, 0), (0, 0, L), (0, L, L)$ and $(0, L, 0)$ and moves with a velocity $\vec{u} = u\hat{i}$. Rain is coming vertically down at constant velocity $\vec{w} = -w\hat{k}$. If the number of drops per unit volume is N , find the number of drops collected by the cardboard as it travels a distance D.

Viewing the same from the cardboard's rest frame (assume relativistic velocities), show that you get the same result for the number of drops collected by the card board.

Solution to Problem 2:

From the figure, it is clear that all the raindrops in the volume (one side of the parallopiped not shown and goes into the paper) ABCD$\times L$ will be collected by the cardboard as it travels a distance $D = AE$. Thus the number of rain drops collected is DL^2N

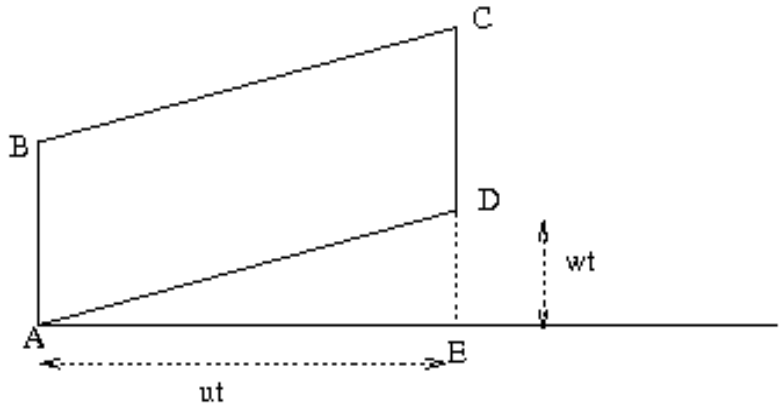

Figure 1.

If we view it from the cardboard's rest frame the velocity of rain drop becomes

$$\vec{w'} = -u\hat{i} - \frac{w}{\gamma}\hat{k}$$

where

$$\gamma = \frac{1}{(1 - u^2/c^2)^{1/2}}$$

The time as seen from the cardboards frame is

$$T' = \gamma(T - \frac{uD}{c^2}) = \frac{T}{\gamma}$$

Volume of the rain collected (depends only on the x-compnent!)is

$$\frac{T}{\gamma}L^2u = \frac{L^2D}{\gamma}$$

The density of rain drops due to Lorentz contraction is $N\gamma$ and thus we get the same answer as viewed from the cardboard's frame,

$$= \frac{L^2D}{\gamma}N\gamma = L^2DN$$

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STUDENT JOURNAL OF PHYSICS

IAPT

Volume 4 Number 5 Oct.-Dec. 2012

Contents

Published on behalf of Indian Association of Physics Teachers
128/2, J.P. Naik Road, Kothurd, Pune-411 038, Phone : 020 25455504

Printed by Acharya Associates, BS-2/3, Acharya Vihar, Bhubaneswar – 751 013