A Numerical Eploration of the Parameter Plane in the Prey - Predator Model

P. K. Thankam[†], P. P. Saratchandran[‡] and K. P. Harikrishnan[‡*]
[†] M. Sc. Student, Department of Physics, The Cochin College, Cochin-682 002, India.
[‡] Department of Physics, The Cochin College, Cochin - 682002, India.

Abstract. We undertake a detailed numerical investigation of the parameter plane of the discrete prey-predator model with the natural death rate of predator in the absence of prey equal to zero. We identify the various dynamical regimes in the parameter plane numerically. Our numerical studies reveal the presence of a region where the asymptotic state deviates from the analytically expected value. We attribute this to the competition between the prey and the predator for survival. We also undertake a dimensional analysis to compute exactly the border line seperating the the periodic and chaotic domains in the parameter plane.

Keywords. Prey-predator Model, Bifurcations, Chaos

1. INTRODUCTION

The problem of the competition between populations of two species is an old one and different models have been proposed to understand its mechanism. The first model in this regard was proposed by the mathematician Voltera [1] in the form of a system of differential equations:

$$\begin{aligned} \frac{dx}{dt} &= ax(1-x) - bxy \\ \frac{dy}{dt} &= -cy + dxy \end{aligned} \tag{1}$$

where $x(t)$ and $y(t)$ represent the population density of prey and predator at time t and a, b, c and d are positive parameters. Here a represents the natural growth rate of the prey in the absence of predators and c represents the natural death rate of predator in the absence of prey. The terms $(-bxy)$ and $(+dxy)$ describe the prey-predator encounters which are favourable to predators and fatal to prey. The dynamics of the prey-predator system has been studied by many authors [2–4].

Note that the above model cannot show any chaotic behavior as it is a two dimensional flow. In fact, discrete models are more reasonable to describe the interaction between species as discussed in detail by May [5]. Such models are more efficient for numerical studies as well and exhibit much richer dynamical structures including chaos, compared to continuous time models. Several such prey-predator and host-parasite models have been formulated and analysed in the past [6–8]. Here we consider the discrete version of the basic prey-predator model corresponding to Eq.(1). There are two versions for this model, one with the natural death rate of predator in the absence of prey $c \neq 0$

*email : $kp\_hk2002@yahoo.co.in$

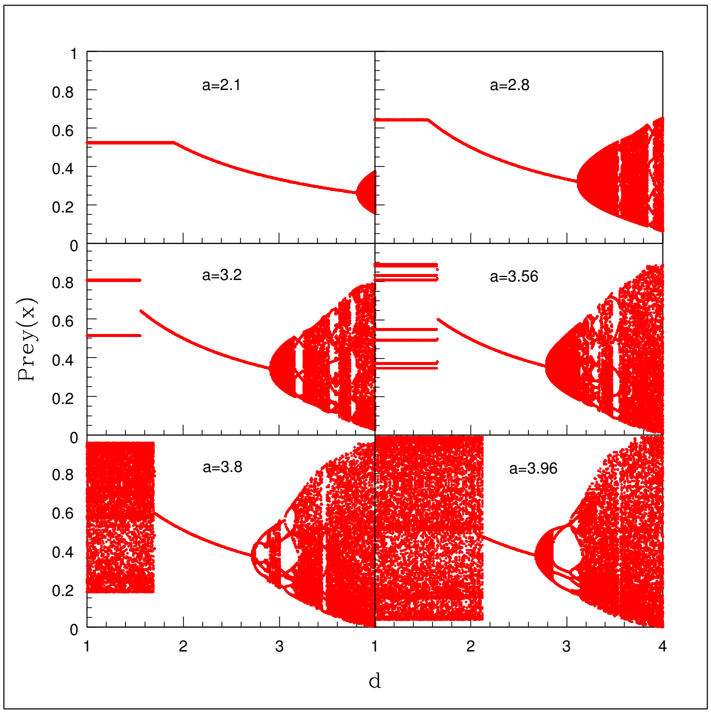

Figure 1. The bifurcation structure of the prey-predator model showing how the population of prey (x) changes with d as the control parameter, for six different values of a. Three different phases can be clearly seen in all cases. The first part indicates period doubling bifurcations to chaos as a varies from 1 to 4 (logistic dynamics), with predator population $y_n \to 0$ for small d. The second phase is a stable fixed point for both x and y which decreases in size as a increases. The last phase represents the post Hopf bifurcation with limit cycles and periodic windows on the way to chaos as d increases. Chaos appears only for sufficiently large values of (a, d).

and the other with $c = 0$. The first version has been studied analytically and numerically by Jing and Yang [9] and Elsadany et al. [10]. The second version with $c = 0$ has been studied in detail by Danca et al. [11] and showed the presence of stable periodic regions, bifurcations and chaos in the model. We undertake a detailed numerical analysis of the complete parameter plane of this model. We identify the exact region of chaos in the parameter plane. Moreover, we show numerically the existence of a small region in the parameter plane where the competition between the predator and prey forces the predator into extinction, beyond the region of extinction obtained analytically. We also undertake a dimensional analysis of the chaotic attractors of the model.

The paper is organised as follows: The next section presents a stability analysis of the model to identify the stable one cycle region. Our main results are discussed in Section 3, where a detailed numerical analysis of the model is undertaken. Conclusions are drawn in Section 4.

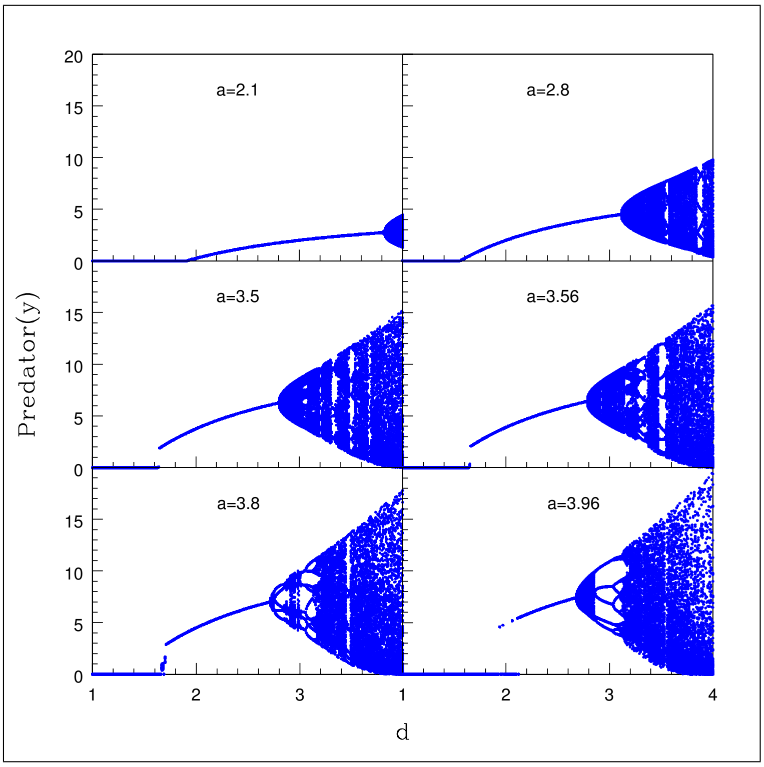

Figure 2. Same as the previous figure, but for predator population y instead of x. Again, three phases can be seen for all a values with the first phase corresponding to predator extinction with $y_n \to 0$. Note that this phase first decreases with a and then increases (that is, extends to larger range of d values) as a becomes large.

2. STABILITY ANALYSIS AND PERIODIC REGIME

The discrete model we consider is given by

$$x_{n+1} = ax_n(1 - x_n) - bx_n y_n$$
$$y_{n+1} = dx_n y_n \tag{2}$$

There are two stable fixed points of the map given by

$$(x_1^*, y_1^*) = (0, 0), (x_2^*, y_2^*) = (\frac{1}{d}, \frac{a}{b}(1 - \frac{1}{d}) - \frac{1}{b})$$

Taking the linearised Jacobian matrix J, the stability of a fixed point can be established by calculating the eigen values λ of J corresponding to the fixed point using the characteristic equation

$$|J - \lambda I| = 0$$

For (x_1^*, y_1^*), we get $\lambda_1 = 0, \lambda_2 = a$. Thus $(0, 0)$ is stable if $a < 1$, irrespective of the value of b and d and both prey and predator vanish asymptotically.

For the second fixed point, $y_2^* > 0$ for $d > \frac{a}{a-1}$. Moreover, the eigen values $\lambda_{1,2}$ corresponding to (x_2^*, y_2^*) are

$$\lambda_{1,2} = (1 - \frac{a}{2d}) \pm \frac{1}{2}\sqrt{(\frac{a}{d} + 2)^2 - 4a} \tag{3}$$

The condition $\lambda_{1,2} < 1$ is satisfied for $d > \frac{a}{a-1}$ and $a > 1$. Thus the condition

$$d = \frac{a}{a - 1} \tag{4}$$

represents a curve in the parameter plane $a - d$ below which the dynamics of prey is governed by the logistic map with a as the control parameter with the population of predator $y_n \to 0$.

Above this curve, the fixed point (x_2^*, y_2^*) becomes stable and one expects a stable one cycle for the co-existence of prey and predator. The region of stability for the fixed point (x_2^*, y_2^*) can be determined by looking at the characteristic equation for J at the fixed point, which can be shown to be

$$P(\lambda) = \lambda^2 - Tr\lambda + Det = 0 \tag{5}$$

where Tr is the trace and Det is the determinant of the Jacobian matrix $J(x_2^*, y_2^*)$ and are given by

$$Tr = 2 - \frac{a}{d} \tag{6}$$

$$Det = a(1 - \frac{2}{d}) \tag{7}$$

If the eigen values λ_i for $J(x_2^*, y_2^*)$ are inside the unit circle in the complex plane, then the fixed point (x_2^*, y_2^*) is locally stable. The necessary and sufficient condition for this are given by

i. $P(1) = 1 - Tr + Det > 0$

ii. $P(-1) = 1 + Tr + Det > 0$

iii. $P(0) = Det < 1$

By substituting the values of Tr and Det, the above 3 conditions can be shown to be equivalent to

$$d > \frac{a}{a-1} \tag{8}$$

$$d > \frac{3a}{a+3} \tag{9}$$

$$d > \frac{2a}{a-1} \tag{10}$$

Thus, the region of stability for the fixed point (x_2^*, y_2^*) is determined by the condition

$$d \in (\frac{a}{a-1}, \frac{2a}{a-1}) \tag{11}$$

The fixed point becomes unstable through a Hopf bifurcation producing a limt cycle. Thus the line of Hopf bifurcation in the parameter plane is given by the condition

$$d = \frac{2a}{a-1} \tag{12}$$

Above this line, the asymptotic state is a limit cycle which may be periodic or quasi periodic depending on the values of a and d. As a and d increases further, the system shows more complex behavior including chaos. We now explore this region of the parameter plane numerically in detail to identify the chaotic regime.

3. NUMERICAL RESULTS

From the analytic results obtained in the previous section, it becomes clear that the value of the parameter b cannot control the asymptotic dynamics; it only determines the position of the attractor in the phase plane. Hence we fix the value of b as 0.2 in all the computations. Since the growth rate of prey in the absence of the predator is governed by the logistic dynamics, we restrict the value of a and d to a maximum of 4. Also, for $a, d < 1$, both x_n and $y_n \to 0$. Hence effectively, the parameter plane $(a - d)$ is restricted within the range $[1 - 4]$. For $a > 4$, the trajectory escapes to ∞. In all our numerical simulations, we use the initial condition x_0, y_0 as $0.63, 0.18$. But we have checked that the results remain unchanged for any initial value in the unit interval $[0, 1]$.

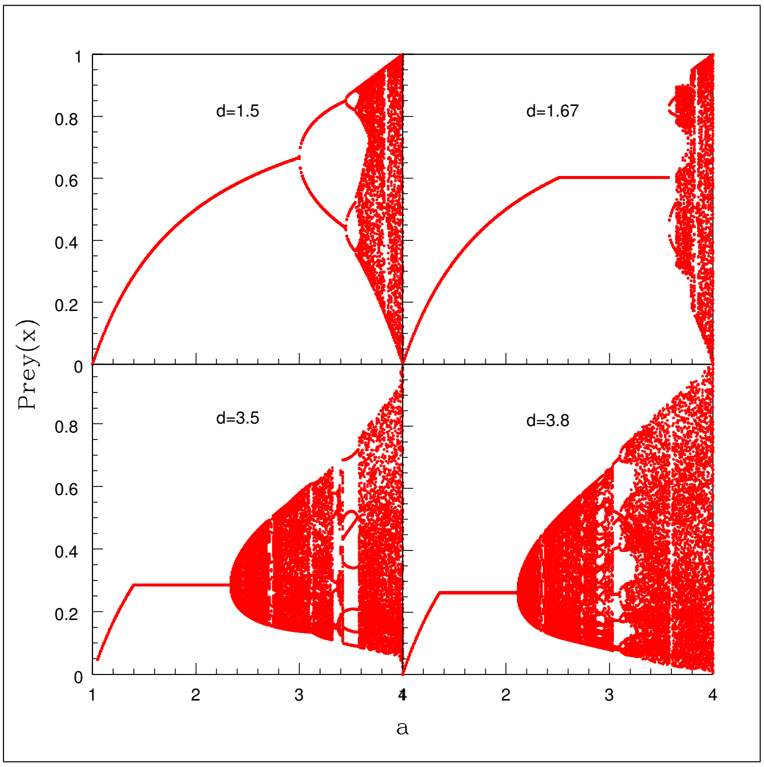

Figure 3. The bifurcation structure for prey population with a as the control parameter for different fixed values of d. When d is sufficiently low (say,$d = 1.5$), the bifurcation structure is clearly that of the logistic map. As d increases, 3 phases can be seen as in Fig. 1 with stable one cycle, Hopf bifurcation and finally chaos. For a small range of d values, (example $d = 1.67$), the dynamics once again re-enters the extinction phase for predator with x values fluctuating with logistic chaos.

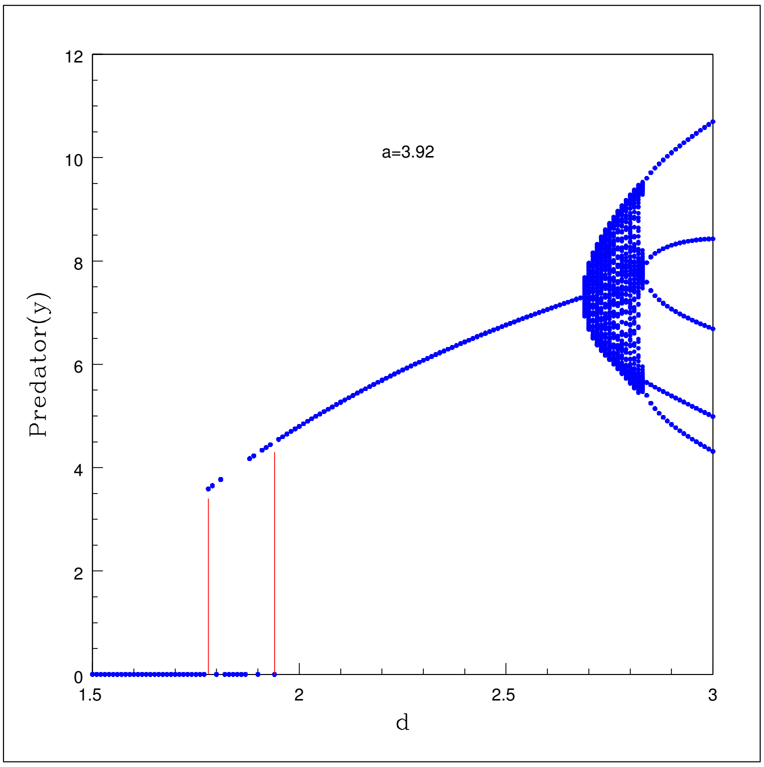

Figure 4. A part of the bifurcation structure for predator with d as the control parameter. There is a small region shown within the two vertical lines where the dynamics sensitively depends on the value of d. The asymptotic state may switch between 0 and a stable state for an infinitesimal change in the control parameter d.

We first compute the bifurcation structure of the prey-predator model for the population of the prey (x) and pedator (y) seperately as a function of d for several fixed values of a in the range $[1, 4]$. The results are shown in Fig. 1 and Fig. 2 respectively for six different values of a. Three seperate phases can be clearly seen in both the figures. The first phase (for small d) indicates period doubling bifurcations to chaos for prey as a increases from 1 to 4. which is just the logistic dynamics in the absence of the predator, $y_n \to 0$. The second phase corresponds to stable co-existence of prey and predator, and its range steadily deceases with a. As the parameter is further increased, the system undergoes a Hopf bifurcation producing a limit cycle (third phase) and finally becomes chaotic at values of a and d as discussed in detail below. Note also that the range of the first phase decreases with a initially, but extends to larger values of d as a increases. This result is completely unexpected, since analytically the domain of predator extinction decreases continuously with a as per Eq.(5). We explore this numerical result in more detail below.

In Fig. 3, we show the bifurcation structure for prey population with a as the control parameter for different fixed values of d. As expected, when d is small, the bifurcation structure is that of the logistic map. When d is sufficiently large, three phases can be clearly seen, namely, predator extinction, stable one cycle and domain of Hopf bifurcation and chaos. For a small range of intermediate d values (for example, $d = 1.67$), the dynamics re-enters the extinction phase for predator where analytically one expects a stable phase. It is clear that there is a small region in the parameter plane where the competition between the predator and prey becomes critical in determining the asymptotic state of the combined system. As the growth rate of prey dominates, the population of the predator is quenched into extinction, stretching the domain of extinction well into the stable region.

Another interesting result we have obtaiined numerically is the identification of a very small regime on the border between the domain of extinction and the stable domain where two stable asymptotic states become riddled depending on the parameter value. In other words, the asymptotic state is sensitively dependent on the parameter value and can switch between two stable states with an infinitesimal perturbation to the parameter. The predator population can switch between stability and extinction while the population of the prey can correspondingly switch between stability and chaotic oscillation. This can be seen from Fig. 4 which is a part of the bifurcation structure for predator with d as the control parameter. There is a small region shown within two vertical lines where the asymptotic state switches between zero and a stable state for an infinitesimal change in the control parameter d. A magnified view of this region is shown in Fig. 5 to make this more clear, along with the corresponding asymptotic state for prey.

As the parameter values are increased, the fixed point becomes unstable through a Hopf bifurcation producing a limit cycle. This is shown in Fig. 6 in the top panel. The limit cycle may be periodic (seen as periodic window in the bifurcation structure) or quasi periodic depending on the parameter values. Both are shown in the bottom panel of Fig. 6.

Our main aim in this work is to identify the domain of chaos exactly in the parameter plane for the prey-predator model. Chaos occurs at critical points for the parameters a and d. We scan the parameter plane numerically increasing a and d in steps of 0.01 and undertake a dimensional

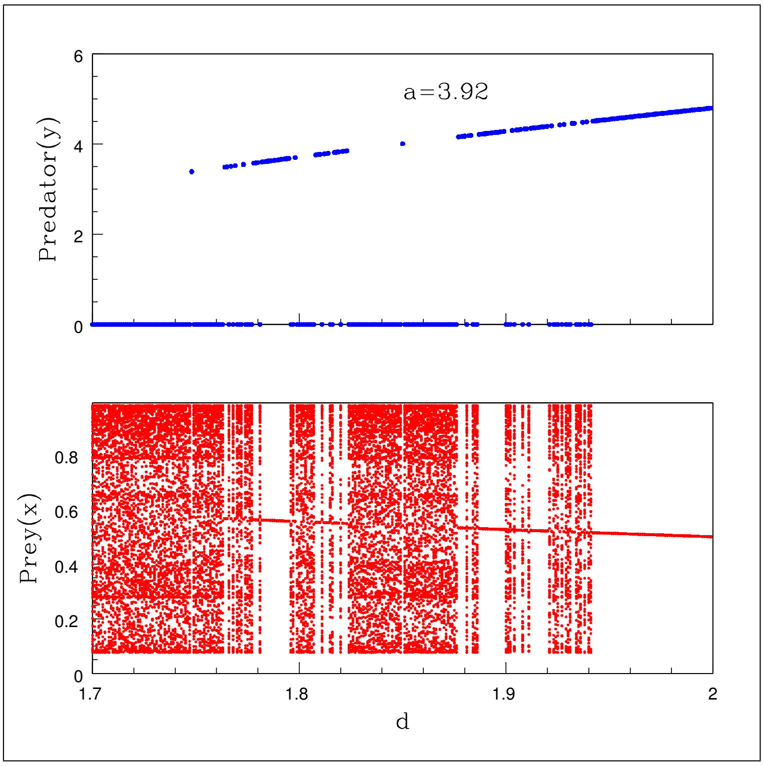

Figure 5. The top panel gives a magnified view of the small region between the two vertical lines in the previous figure. The lower panel shows the corresponding values of x for that region. Note that as y switches between 0 and a stable state, x switches between logistic chaos and a stable state.

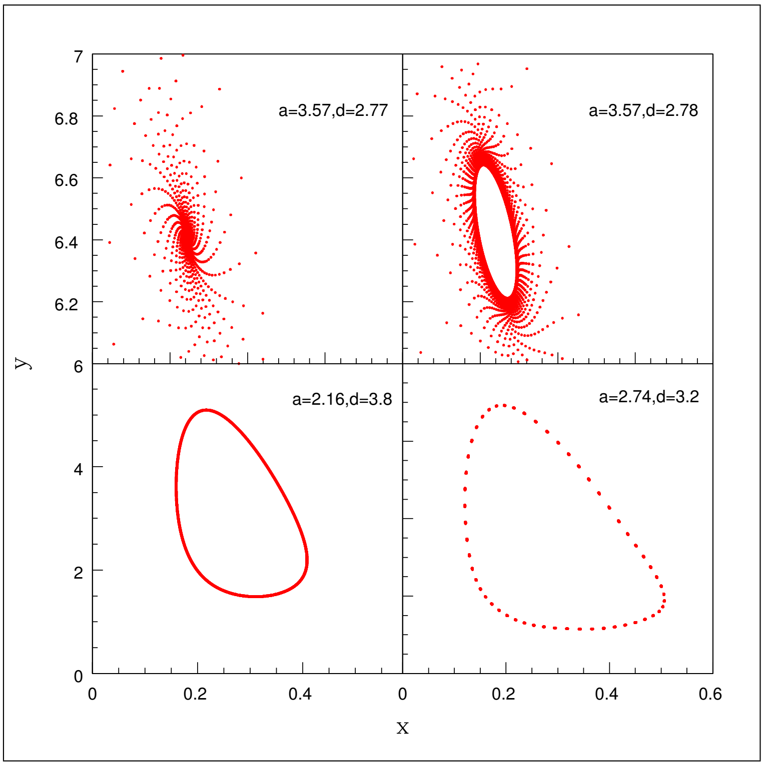

Figure 6. The top panel shows the nature of the attractor just before and after the Hopf bifurcation. The lower panel shows a quasi periodic limit cycle and a periodic window.

analysis to detect chaos. We use the modified box counting algorithm [12] to compute the correlation dimension D_2 by generating the time series. By using this scheme, we are able to locate the domain of chaos exactly in the parameter plane. In Fig. 7, we show the chaotic attractors for four different sets of parameter values (a, d). The corresponding D_2 values are shown in Fig. 8.

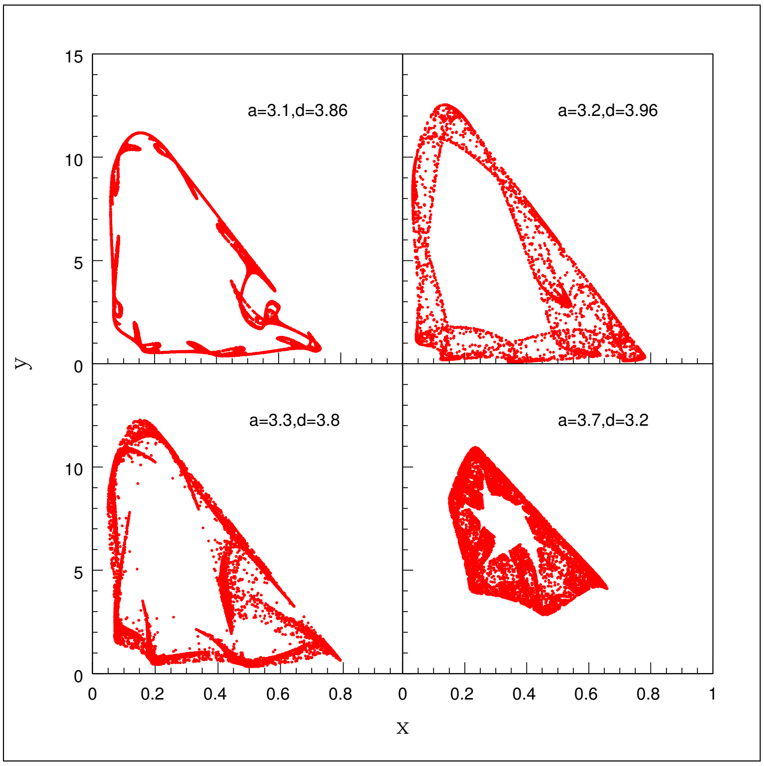

Figure 7. The chaotic attractors of the prey-predator model for four different parameter values.

To summarise, the different dynamical regimes of the prey-predator model obtained from our numerical analysis are shown in Fig. 9. The domain of chaos is denoted as region IV. The region (denoted as I) below the dashed line corresponds to extinction of predator and logistic dynamics for prey as a increases. The dotted line is the analytic curve (Eq.(5)) above which the second fixed point (x_2^*, y_2^*) should become stable. But numerically we find that the domain of extinction encroaches into the stable domain for lager a values. Thus, there is a region between the two lines (denoted V) where analytical and numerical results disagree and the predator is forced into extinction in competition with the prey. There is also a smaller region within this denoted by triangles, where the asymptotic state of the system depends sensitively on the parameter values.

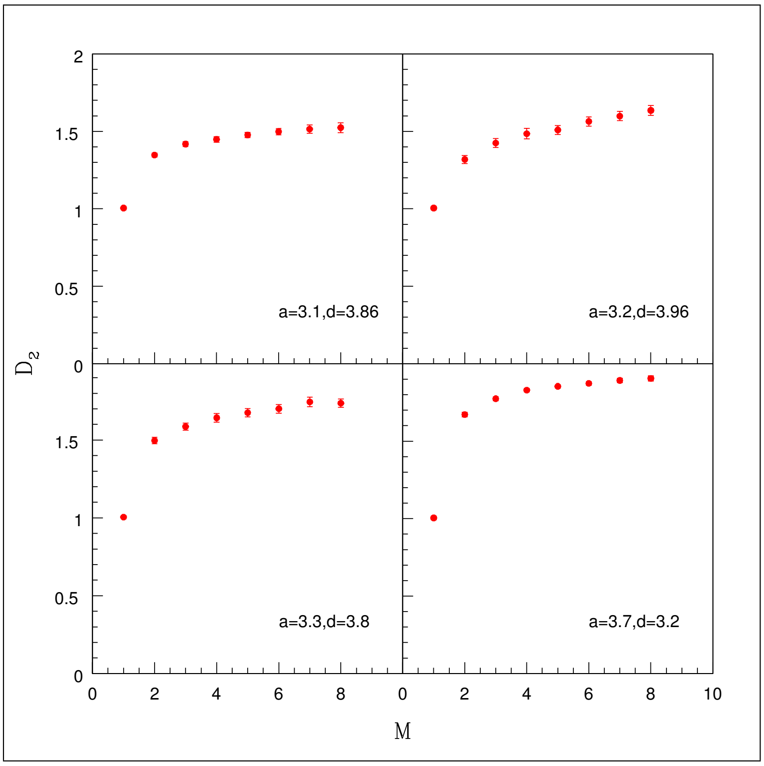

Figure 8. The correlation dimension D_2 of the four chaotic attractors in the previous figure as a function of the embedding dimension M.

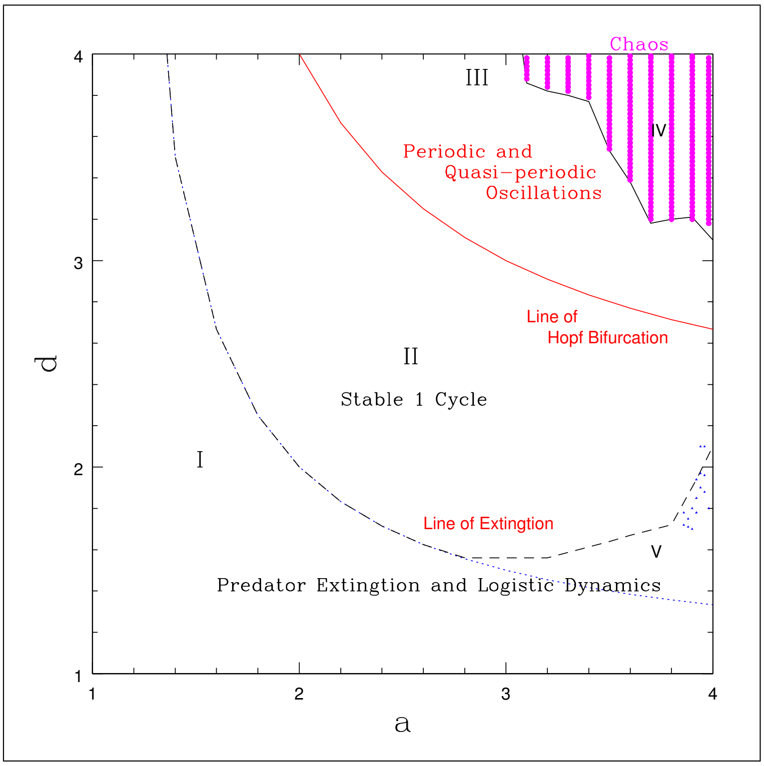

Figure 9. The complete parameter plane of the prey-predatoe model showing different domains of dynamics. The dotted line marks the stability of the second fixed point of the model. The dashed line which coincides with the dotted line for the major part, but bifurcates from it for larger a values represents the line of extinction of the predator. In the region below this line, denoted I, $y_n \rightarrow 0$ and the model displays logistic dynamics. The region II above this line represents the co-existence of predator and prey with stable one cycle. The solid line represents the transition from stable one cycle to periodic and quasi periodic oscillations by way of Hopf bifurcation. The region IV with solid vertical lines at the top right corner is where the dynamics turns chaotic. The region V where the dashed line bifurcates from the dotted line represents the competition between predator and prey and the predator is forced into extinction. There is a small part inside this, denoted by scattered triangles, where the asymptotic state of the system depends sensitively on the parameter values.

4. CONCLUSION

The prey-predator model is one of the most studied models in the context of deterministic chaos. Different versions of this model have been extensively studied by many authors in the past, both analytically and numerically. In this study, we numerically explore the parameter plane of the basic prey-predator model to identify various dynamical regimes, especially the chaotic regime. We are able to locate for the first time the exact domain of occurance of chaos in the parameter plane through a dimensional analysis. We compute the fractal dimension of the chaotic attractors for typical parameter values using a modified box counting scheme. Another interesting result we have obtained is the identification of a domain in the parameter plane where the asymptotic state deviates numerically from the analytically expected value and also a small region where the asymptotic state depends sensitively on the parameter values.

5. ACKNOWLEDGEMENTS

The work was done as part of the M. Sc project of PKT under the guidance of KPH. PKT acknowledges the support from the faculty and friends in the Department of Physics, The Cochin College.

References

[1] V. Voltera, *Opere Matematiche: Memorie e Note, Vol.5*, Acc. Naz. dei Lincei, Roma, 1962
[2] J. Hainzl, *SIAM J. Appl. Mathematics* **48**, 170(1988).
[3] X. He, *J. Mathematical Analysis & Applications* **198**, 355(1996).
[4] J. D. Murray, *Mathematical Biology*, (Springer, Berlin, 1998).
[5] R. M. May, *J. Theor. Biology* **51**, 511(1975).
[6] Z. J. Jing, Y. Chang and B. Guo, *Chaos, Solitons & Fractals* **21**, 701(2004).
[7] P. V. Ivanchikov and L. V. Nedorezov, *Comp. Ecology and Software* **1(2)**, 86(2011).
[8] A. A. Elsadany, *Comp. Ecology and Software* **2(2)**, 124(2012).
[9] Z. J. Jing and J. Yang, *Chaos, Solitons & Fractals* **27**, 259(2006).
[10] A. A. Elsadany, H. A. El-Metwally, E. M. Elabbasy and H. N. Agiza, *Comp. Ecology and Software* **2(3)**, 169(2012).
[11] M. Danca, S. Codreanu and B. Bako, *J. Biological Physics* **23**, 11(1997).
[12] K. P. Harikrishnan, R. Misra and G. Ambika, *Comm. Nonlinear Sci. Numer. Simulations* **17**, 263(2012).