Studying the Puzzle of the Pion Nucleon Sigma Term

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Abstract. In this paper I investigate the flavor dependence of the pion nucleon sigma term ($\sigma_{\pi N}$) for the $N_f = 2$, $N_f = 2 + 1$, and $N_f = 2 + 1 + 1$ cases, where N_f is the number of flavors. I calculate $\sigma_{\pi N}$ using the Hellmann-Feynman method which uses results of lattice quantum chromodynamics (LQCD). I use the expansion from Baryon Chiral Perturbation Theory as my nucleon mass fitting equation. I extrapolate the data to $a \to 0$, where a is the spacing of the lattice in LQCD, and apply the constraint that data must meet the condition $M_\pi L > 3.8$ to avoid finite volume effects, where M_π is the pion mass and L is the length of the lattice in LQCD. My results shed light on the recent disparity between values of $\sigma_{\pi N}$ calculated using different methods.

1. INTRODUCTION

The search for dark matter has seen a surge of interest in recent years with the hope of finding physics beyond the standard model. All current experimental searches rely on dark matter particles interacting with nucleonic matter, i.e. protons and neutrons. One leading candidate for a dark matter particle is the neutralino, which is predicted by the theory of super-symmetry [1]. In order to constrain experiments searching for the neutralino, the cross section of interaction with nucleons must be known. The pion-nucleon sigma term ($\sigma_{\pi N}$), which is a fundamental parameter in the theory of quantum chromodynamics (QCD), is used to calculate this cross section [2]. It was originally calculated by phenomenological methods but recently has been calculated using methods involving LQCD. There is a disparity between the two methods however, with $\sigma_{\pi N}$ being significantly lower using the latter method [3]. This disparity is large enough to cause concern in the dark matter community as experiments would need to be changed accordingly.

One method of calculating $\sigma_{\pi N}$ using LQCD data is called the Hellman-Feynman (HF) method. The HF theorem relates $\sigma_{\pi N}$ to the nucleon mass (M_N) dependence on the quark mass (m_q) [4]. The HF theorem can also relate $\sigma_{\pi N}$ to the nucleon mass dependence on the pion mass (M_π) as $M_\pi^2 = m_q$. From this point on, M_π and m_q will be used interchangeably with this understanding. The HF theorem is defined in Eq. 1.

$$\sigma_{\pi N} = m_q \frac{\partial}{\partial m_q} M_N(m_q) = M_\pi^2 \frac{\partial}{\partial M_\pi^2} M_N(M_\pi^2) \tag{1}$$

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LQCD is used to calculate the nucleon mass from a given quark mass (quark masses need not be physical). These data points are in turn used to determine the nucleon mass dependence on the quark mass the HF theorem requires to calculate $\sigma_{\pi N}$. It does so by simulating the dynamics inside the nucleon. Nucleons are composed of three valence quarks, but from the Heisenberg Uncertainty Principle, $\Delta E \Delta t \geq \hbar 2$, we know that quark-antiquark pairs can be created and annihilated from the vacuum. Heavier quarks will be created for shorter periods of time and therefore will have a smaller effect on the internal dynamics of the nucleon. It is common in LQCD simulations to assume that only the two and three lightest flavors (up, down, strange) of quarks contribute to the dynamics and that contributions from the heavier flavors (charmed, top, and bottom) can be ignored. In this paper I present results of $\sigma_{\pi N}$ calculated from data that included the two lightest flavors ($N_f = 2$), three lightest flavors ($N_f = 2 + 1$), and four lightest flavors ($N_f = 2 + 1 + 1$) to see if the heavier quarks have a significant contribution or if they can be safely ignored in further simulations.

2. LATTICE QCD

2.1 *Overview*

QCD is the theory that describes how quarks and gluons interact via the strong force. At high energies, i.e. particle accelerators, perturbation theory can be used to perform precise calculations. At low energies however, i.e. inside a nucleon, perturbation theory fails and calculations can no longer be done. Lattice QCD is a fully non-perturbative formulation of QCD that can perform calculations at any energy [5].

LQCD works by putting spacetime on a grid as seen in Fig. 1. The quark field exists on the lattice sites and the gluon field exists on the lattice bonds connecting neighboring sites. Each lattice has three important properties, the lattice spacing a, the box length L, and the quark mass m_q. LQCD is developed such that if the following limits are taken, $\lim a \rightarrow 0$, $\lim L \rightarrow \infty$, $\lim m_q \rightarrow m_{q,physical}$, physical QCD is recovered. Because the universe we live in is continuous and infinite however, any calculations done on a lattice inherently contain systematic errors. The two major sources of systematic error are finite volume effects (L) and lattice spacing effects (a).

2.2 *Finite Volume Effects*

Finite volume effects occur when L is small compared to the wavelength of the quarks on the lattice. Instead of comparing L to the wavelength, it is usually compared to the *mass* of the quarks. To see how this can simplify quantifying finite volume effects, we look at the DeBroglie wave equation $\lambda = hp$. Because less mass implies less momentum, less mass implies a larger wavelength. So the smaller the quark mass, the larger the box must be. A useful quantity to look at in judging the magnitude of the finite volume effects is therefore the product of the pion mass and the box length $M_\pi L$. If this quantity is large enough, it is reasonable to ignore the systematic error. If it is small, steps must be taken to extrapolate the calculations to $L \rightarrow \infty$ and account for the finite volume

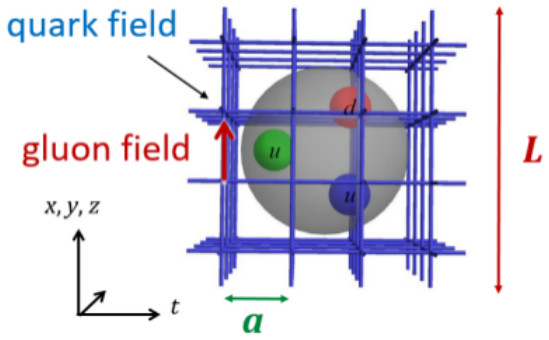

Figure 1. Image of a proton on a typical lattice in LQCD. The lattice spacing a is in green and the box length L is in red. The image is reproduced with the permission of Professor H. Lin.

effects.

2.3 *Lattice Spacing Effects*

Finite lattice spacing effects occur when a is too large to properly simulate strong force dynamics. The value of a necessary is less dependent on m_q than L. A standard value of a generally accepted to limit the size of finite lattice spacing effects is $a < 0.1$ fm. The leading corrections in the lattice spacing effects that remain are typically $O(a^2)$ [5]. As systematic errors are unique to each simulation, results can be extrapolated to $a \to 0$ by adding a term $c_j a^2$ to the fitting equation where j indicates what collaboration each data point was calculated by. The c_j terms are then treated as fitting parameters.

3. METHOD OF CALCULATION

3.1 *Applying the Hellmann-Feynman Theorem*

The HF methods requires a functional relationship between the nucleon mass and quark mass. To achieve the necessary functional relationship, I use the expansion taken from Baryon Chiral Perturbation Theory (BχPT) [6]. The first several terms of the expansion in the pion mass can be seen in Eq. 2.

$$M_N(M_\pi^2) = M_0 - 4C_1 M_\pi^2 + \frac{1}{2}\bar{\alpha} M_\pi^4 + \frac{C_1}{8\pi^2 f_\pi^2} M_\pi^4 \ln \frac{M_\pi^2}{M_0} \tag{2}$$

The terms M_0, C_1, and $\bar{\alpha}$ are low energy constants (LECs) that must be determined before calculating $\sigma_{\pi N}$. Once they are known, the HF theorem allows for a straightforward calculation of $\sigma_{\pi N}$ through a simple analytic derivative given by Eq. 3.

$$\sigma_{\pi N} = -4C_1 M_\pi^2 + \bar{\alpha} M_\pi^4 + \frac{C_1}{4\pi^2 f_\pi^2} M_\pi^4 \ln \frac{M_\pi^2}{M_0^2} + \frac{C_1}{8\pi^2 f_\pi^2} M_\pi^4 \tag{3}$$

3.2 *Determining the Low Energy Constants*

The LECs are determined by fitting Eq. 2 to the nucleon mass data generated by LQCD using varying pion masses. All sources of error are small enough compared to the error in the values of M_N and are therefore deemed negligible. Additionally, errors in M_N are assumed to be uncorrelated. To ensure that finite volume effects were negligible, points that did not satisfy $M_\pi L > 3.8$ did not enter the fit. Points were extrapolated to $a \to 0$ by including a term of the form $c_j a^2$ in the fit for each collaboration data was taken from. Three c_j terms were added in the $N_f = 2$ case and two c_j terms were added in both the $N_f = 2 + 1$ and $N_f = 2 + 1 + 1$ case. The final χ^2 function that I minimize is

$$\chi^2 = \sum_{i=1} \frac{M_N(M_\pi^2) + c_j a^2 - d_i(M_\pi^2)}{\sigma_i}, \tag{4}$$

where $d_i(M_\pi^2)$ are the LQCD data points for M_N with associated uncertainties σ_i. The common fitting parameters for all three fits include M_0, C_1, and $\bar{\alpha}$. A good fit will have χ^2/dof $\cong 1$, where dof is short for the degrees of freedom in the fit and is defined as the number of data points (d_i) minus the number of fitting parameters (i.e. M_0, C_1, $\bar{\alpha}$, c_j). Uncertainties in the fit parameters, nucleon mass, and $\sigma_{\pi N}$ were determined using the standard jackknife procedure described in [7]. All values will be given in the form mean(stdev). As an example, 7.92(13) shows that the mean value is 7.92 with an associated standard deviation is 0.13.

4. RESULTS

For the fit using $N_f = 2$ LQCD collaboration data, seven points were taken from the Mainz collaboration [8], six points were taken from the RQCD collaboration [9], and seven points were taken from the ETM collaboration [10]. The three extrapolation parameters (c_j of Eq. 4), can be found in Table 1. The c_{LQCD} and c_{RQCD} terms are consistent with zero while the c_{Mainz} term is not. This shows that the systematic error introduced in the ETM and RQCD collaborations were similar in magnitude and thus a non-zero extrapolation was necessary for the Mainz data points. For the $N_f = 2 + 1$ fit, nine points were taken from the LHP collaboration [11] and five points were taken from the NME collaboration [12]. The extrapolation parameters c_{LHP} and c_{NME} are not consistent with zero as seen in Table 1. For the $N_f = 2 + 1 + 1$ fit, fifteen points were taken from the ETM collaboration

[13] and six points were taken from the PNDME collaboration [14]. Like the $N_f = 2 + 1$ case, the extrapolation parameters c_{ETMC} and c_{PNDME} are not consistent with zero as seen in Table 1.

2*N_f = 2	c_{RQCD}	$-0.12(15)$
	c_{ETMC}	0.15(14)
	c_{Mainz}	0.55(19)
2*N_f = 2+1	c_{LHP}	0.124(9)
	c_{NME}	$-0.166(4)$
2*N_f = 2+1+1	c_{ETMC}	0.136(4)
	c_{PNDME}	$-0.042(25)$

Table 1. Extrapolation parameter values ($a \to 0$) for the N_f = 2, 2+1, 2+1+1 fits.

The fit for the $N_f = 2$ case can be seen in Fig. 2. The large χ^2/dof can be explained by analyzing the contribution of each individual data point to the total value. In this fit, three data points contributed to more than 50% of the total value. From this it is seen that using Eq. 2 as the fitting equation is appropriate, and the systematic error in the three data points in question was underestimated. The fit for the $N_f = 2 + 1$ case can be found in Fig. 3. Although the χ^2/dof is smaller than in the $N_f = 2$ case, it is still not low enough to be considered a good fit. Similar to the previous case however, three data points accounted for over 50% of the value. This leads to the same conclusion that the systematic error in those data points were underestimated. The fit for the $N_f = 2 + 1 + 1$ case can be seen in Fig. 4. The χ^2/dof is within the range to indicate a good fit. This shows that errors in all data points have appropriate errors associated with them.

N_f	M_0 [GeV]	C_1 [GeV^{-1}]	$\bar{\alpha}$ [GeV^{-3}]	$\sigma_{\pi N}$ [MeV]	$\chi^2 dof$
2	0.908(4)	$-0.55(6)$	$-5.4(1.8)$	40(4)	4.76
2+1	0.901(23)	$-0.26(18)$	11(10)	25(11)	2.04
2+1+1	0.916(18)	$-0.56(4)$	$-7.5(9)$	40(3)	1.31

Table 2. Results for BχPT fits to N_f = 2, 2+1, 2+1+1 nucleon mass data.

The values of $\sigma_{\pi N}$ for the $N_f = 2$ and $N_f = 2 + 1$ cases, seen in Table 2, agree with values produced by [6] within errorbars. Comparing the values of $\sigma_{\pi N}$ for all three cases, we see that the mean values for the $N_f = 2$ and $N_f = 2 + 1 + 1$ cases are equal with similar error bars. The mean value for the $N_f = 2 + 1$ case is significantly smaller comparatively, but has a standard deviation of 40% the mean value. Because of this large error, the value still agrees with the $N_f = 2$ case within error bars and is just over one standard deviation away from agreeing with the $N_f = 2 + 1 + 1$ case within error bars. Furthermore, comparing the values of the fitting parameters it is seen that the values of M_0 for the three cases are not statistically different. The values of C_1 and $\bar{\alpha}$ for the $N_f = 2$ and $N_f = 2 + 1 + 1$ cases are indistinguishable while the values for the $N_f = 2 + 1$ case disagree.

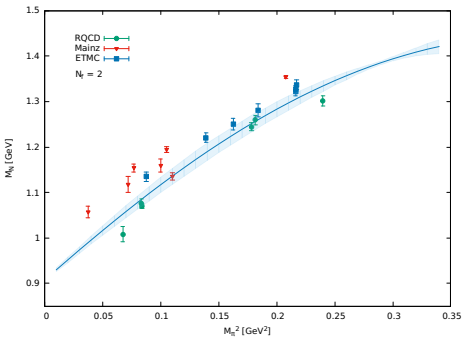

Figure 2. N_f=2 flavor fit of nucleon mass vs. pion mass squared. Data points from RQCD are in green, from Mainz are in red, and from ETMC are in blue. The shaded blue region is the uncertainty in the M_N calculations while the solid dark blue line is the mean value of M_N.

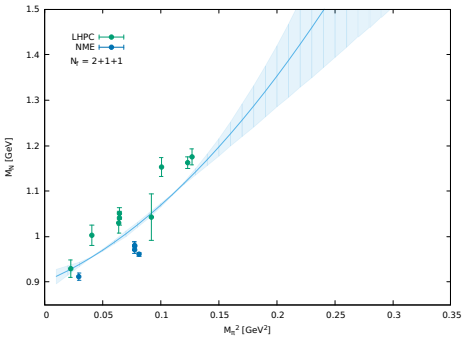

Figure 3. $N_f = 2 + 1$ flavor fit of nucleon mass vs. pion mass squared. Data points from NME are in blue and points from LHPC are in green.

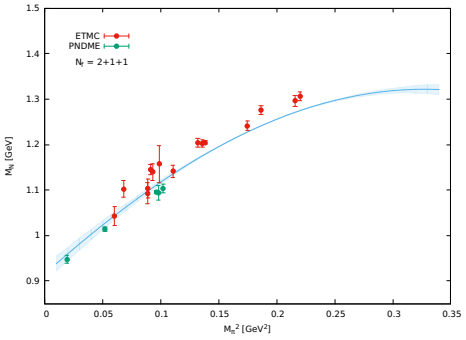

Figure 4. $N_f = 2 + 1 + 1$ flavor fit of nucleon mass vs. pion mass squared. Data points from ETMC are in red and points from PNDME are in green.

However, the error in C_1 and $\bar{\alpha}$ for the $N_f = 2 + 1$ case are, again large, with values of 70% and 90% of the mean respectively. One possibility for the large error in the $N_f = 2 + 1$ case is the small number of available data points compared to the other cases. The $N_f = 2$ and $N_f = 2 + 1 + 1$ had twenty data points that met the fit requirements while the $N_f = 2 + 1$ case had only fourteen points that met the fit requirements. Because of the large error in the parameters for the three flavor case, I cannot conclude the value of $\sigma_{\pi N}$ is statistically different from the two and four flavor case. The fits therefore show that the values of $\sigma_{\pi N}$ for the three cases are not statistically different and there is no apparent flavor dependence.

5. SUMMARY AND CONCLUSION

In this work I collected data from various collaborations generated using lattice QCD for the two flavor, three flavor, and four flavor cases. The data needed to meet the requirement that $M_\pi L > 3.8$ to assure finite volume effects could be safely ignored. Terms of the form $c_j a^2$ were added to the fitting equation to account for lattice spacing effects. This data was fitted to an expansion of the nucleon mass in terms of the pion mass developed from Baryon Chiral Perturbation Theory. Once the low energy constants were determined, I applied the Hellmann-Feynman theorem to the fitting equation in order to calculate $\sigma_{\pi N}$. Comparing the values of $\sigma_{\pi N}$ for the three cases, it is seen that they are not statistically different. This shows that after a first level analysis, $\sigma_{\pi N}$ has no significant dependence on the number of flavors included in the LQCD simulations. The inclusion of heavier quarks can therefore not account for the disparity in values calculated by phenomenological methods and LQCD methods.

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