Lepton Family Symmetries for Neutrino Masses and Mixing

Ernest Ma
Physics Department
University of California
Riverside, CA 92521, USA

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Some Basics

$$\mathcal{M}_l = U_L^l \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} (U_R^l)^\dagger ,$$

$$\mathcal{M}_\nu^D = U_L^\nu \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} (U_R^\nu)^\dagger ,$$

$$\mathcal{M}_\nu^M = U_L^\nu \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} (U_L^\nu)^T ,$$

$$U_{l\nu} = (U_L^l)^\dagger U_L^\nu = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix}$$

$$\simeq \begin{pmatrix} 0.83 & 0.56 & < 0.2 \\ -0.39 & 0.59 & -0.71 \\ -0.39 & 0.59 & 0.71 \end{pmatrix}$$

$$\simeq \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix} \sim (\eta_8, \eta_1, \pi^0)$$

[Harrison, Perkins, Scott (2002)]

How can the HPS form of $U_{l\nu}$ be derived from a symmetry? The difficulty comes from the fact that any symmetry defined in the basis $(\nu_e, \nu_\mu, \nu_\tau)$ is automatically applicable to (e, μ, τ) in the complete Lagrangian. Usually one assumes the canonical seesaw mechanism and studies

$$\mathcal{M}_\nu = -\mathcal{M}_\nu^D \mathcal{M}_N^{-1} (\mathcal{M}_\nu^D)^T$$

in the basis where $\mathcal{M}_l$ is diagonal. But the symmetry apparent in $\mathcal{M}_\nu$ is often incompatible with a diagonal $\mathcal{M}_l$ with eigenvalues m_e, m_μ, m_τ.

Consider just 2 families. Suppose we want maximal $\nu_\mu - \nu_\tau$ mixing, then

$$\mathcal{M}_\nu = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a+b & 0 \\ 0 & a-b \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

This seems to require the exchange symmetry $\nu_\mu \leftrightarrow \nu_\tau$, but since (ν_μ, μ) and (ν_τ, τ) are $SU(2)_L$ doublets, we must also have $\mu \leftrightarrow \tau$ exchange.

We now have the option of assigning μ^c and τ^c. If $\mu^c \leftrightarrow \tau^c$ exchange is also assumed, then

$$\mathcal{M}_l = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} A+B & 0 \\ 0 & A-B \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Hence $U_{l\nu} = (U_L^l)^\dagger U_L^\nu = 1.$

If μ^c and τ^c do not transform under this exchange, then

$$\mathcal{M}_l = \begin{pmatrix} A & B \\ A & B \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2(A^2+B^2)} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c & s \\ -s & c \end{pmatrix},$$

where $c = A/\sqrt{A^2+B^2}$, $s = B/\sqrt{A^2+B^2}$.

Again $U_{l\nu} = (U_L^l)^\dagger U_L^\nu = 1$.

Some Discrete Symmetries

solid	faces	vert.	Plato	Hindu	sym.
tetrahedron	4	4	fire	Agni	A_4
octahedron	8	6	air	Vayu	S_4
cube	6	8	earth	Prithvi	S_4
icosahedron	20	12	water	Jal	A_5
dodecahedron	12	20	quintessence	Akasha	A_5

Compare this to today's TOE, i.e. string theory. There are 5 consistent theories in 10 dimensions: Type I is dual to Heterotic $SO(32)$, Type IIA is dual to Heterotic $E_8 \times E_8$, and Type IIB is self-dual.

- Plato inferred the existence of the fifth element (quintessence) from the mismatch of the 4 known elements (fire, air, water, earth) with the 5 perfect geometric solids.

- Glashow, Iliopoulos, and Maiani inferred the existence of the fourth quark (charm) from the mismatch of the 3 known quarks (up,down,strange) with the 2 charged-current doublets $(u, d\cos\theta_C + s\sin\theta_C)$ and $(?, -d\sin\theta_C + s\cos\theta_C)$.

Question: What sequence has $\infty, 5, 6, 3, 3, 3, \ldots$?
Answer: Perfect geometric solids in 2,3,4,5,6,7,...
dimensions. In 4 space dimensions, they are:

solid	composition	faces	vertices
4-simplex	tetrahedron	5	5
4-crosspolytope	tetrahedron	16	8
4-cube	cube	8	16
600-cell	tetrahedron	600	120
120-cell	dodecahedron	120	600
24-cell	octahedron	24	24

Tetrahedral Symmetry A_4

For 3 families, we should look for a group with a $\underline{3}$ representation, the simplest of which is A_4, the group of the even permutation of 4 objects.

class	n	h	χ_1	$\chi_{1'}$	$\chi_{1''}$	χ_3
C_1	1	1	1	1	1	3
C_2	4	3	1	ω	ω^2	0
C_3	4	3	1	ω^2	ω	0
C_4	3	2	1	1	1	–1

$$\omega = \exp(2\pi i/3) = -1/2 + i\sqrt{3}/2$$

Multiplication rule:

$$\underline{3} \times \underline{3} = \underline{1}(11 + 22 + 33) + \underline{1}'(11 + \omega^2 22 + \omega 33)$$

$$+\underline{1}''(11 + \omega 22 + \omega^2 33) + \underline{3}(23, 31, 12) + \underline{3}(32, 13, 21).$$

Note that $\underline{3} \times \underline{3} \times \underline{3} = \underline{1}$ is possible in A_4,
i.e. $a_1 b_2 c_3$ + permutations,
and $\underline{2} \times \underline{2} \times \underline{2} = \underline{1}$ is possible in S_3,
i.e. $a_1 b_1 c_1 + a_2 b_2 c_2$.

$\mathcal{M}_l$,$\mathcal{M}_\nu$,$U_{l\nu}$

Consider $(\nu_i, l_i) \sim \underline{3}$ under A_4, then $\mathcal{M}_\nu$ is of the form

$$\mathcal{M}_\nu = \begin{pmatrix} a+b+c & f & e \\ f & a+\omega b+\omega^2 c & d \\ e & d & a+\omega^2 b+\omega c \end{pmatrix}$$

where a comes from $\underline{1}$, b from $\underline{1}'$, c from $\underline{1}''$, and (d,e,f) from $\underline{3}$.

In this basis, $\mathcal{M}_l$ is generally not diagonal, but under A_4, there are two interesting cases.

(I) Let $l^c_i \sim \underline{1}, \underline{1}', \underline{1}''$, then with $(\phi^0_i, \phi^-_i) \sim \underline{3}$,

$$\mathcal{M}_l = \begin{pmatrix} h_1 v_1 & h_2 v_1 & h_3 v_1 \\ h_1 v_2 & h_2 \omega v_2 & h_3 \omega^2 v_2 \\ h_1 v_3 & h_2 \omega^2 v_3 & h_3 \omega v_3 \end{pmatrix}$$

$$= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix} \sqrt{3} v$$

for $v_1 = v_2 = v_3 = v$.

(II) Let $l_i^c \sim \underline{3}$, but $(\phi_i^0, \phi_i^-) \sim \underline{1}, \underline{1}', \underline{1}''$, then

$$\mathcal{M}_l = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}.$$

In either case, it solves the fundamental theoretical problem of having a symmetry for the neutrino mass matrix even though the charged-lepton mass matrix has three totally different eigenvalues.

To proceed further, the 6 parameters of $\mathcal{M}_\nu$ must be restricted, from which $U_{l\nu}$ may be obtained.

$$U_L^\dagger \mathcal{M}_\nu U_L^* = \mathcal{M}_\nu^{(e,\mu,\tau)} = U_{l\nu} \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} (U_{l\nu})^T,$$

where (I)

$$U_L = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix},$$

and (II) $U_L = 1$.

Neutrino Mass Models

Using (I), the first two proposed A_4 models start with only $a \neq 0$, yielding thus 3 degenerate neutrino masses. In Ma/Rajasekaran(2001), the degeneracy is broken softly by $N_i N_j$ terms, allowing b, c, d, e, f to be nonzero. In Babu/Ma/Valle(2003), the degeneracy is broken radiatively through flavor-changing supersymmetric scalar lepton mass terms. In both cases, $\theta_{23} \simeq \pi/4$ is predicted. In BMV03, maximal CP violation in $U_{l\nu}$ is also predicted.

Consider the case $b = c$ and $e = f = 0$, then

$$\mathcal{M}_\nu^{(e,\mu,\tau)} = \begin{pmatrix} a+2d/3 & b-d/3 & b-d/3 \\ b-d/3 & b+2d/3 & a-d/3 \\ b-d/3 & a-d/3 & b+2d/3 \end{pmatrix}$$

$$= a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \frac{d}{3} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

$$= U_{l\nu} \begin{pmatrix} a-b+d & 0 & 0 \\ 0 & a+2b & 0 \\ 0 & 0 & -a+b+d \end{pmatrix} (U_{l\nu})^T$$

and

$$U_{l\nu} = \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}$$

i.e. tribimaximal mixing would be achieved.

However, although $b \neq c$ would allow $U_{e3} \neq 0$, the assumption $e = f = 0$ and the bound $|U_{e3}| < 0.16$ imply $0.5 < \tan^2\theta_{12} < 0.52$.

Experimentally, $\tan^2\theta_{12} = 0.45 \pm 0.05$.

Models based on (I) with $d \neq 0$ and $e = f = 0$:

- Ma(2004)
- Altarelli/Feruglio(2005-1/2): $b = c = 0$.
- Ma(2005-1): $a = 0, \; [b = c]$.
- Babu/He(2005): $b = c, \; d^2 = 3b(b - a)$.
- Zee(2005): $[b = c]$.
- Ma(2005-5): $b = c$.

Particle content of models based on (I):

A_4(I)	ϕ^+, ϕ^0	N	ξ^{++}, ξ^+, ξ^0	χ^0	SUSY
MR01	1,3	3	—	—	no
BMV01	1,1	3	—	3	yes
M04	3	-	$1, 1', 1'', 3$	—	no
AF05-1	1,1	-	—	1,3,3	no
M05-1	3	-	$1', 1'', 3$	—	no
BH05	1,1	3	—	1,1,3,3	yes
Z05	3	-	—	—	no
M05-5	1,1	3	—	3	yes
AF05-2	1,1	-	—	1,1,1,3,3,3,3	yes

M05-5 [Ma, hep-ph/0511133]

$$\mathcal{M}_D = U_L^\dagger \begin{pmatrix} m_D & 0 & 0 \\ 0 & m_D & 0 \\ 0 & 0 & m_D \end{pmatrix}, \quad \mathcal{M}_N = \begin{pmatrix} A & 0 & 0 \\ 0 & B & C \\ 0 & C & B \end{pmatrix}$$

imply $e = f = 0$ and $b = c$. To obtain this $\mathcal{M}_N$, consider

$$W = \frac{1}{2} m_N (N_1^2 + N_2^2 + N_3^2) + f N_1 N_2 N_3$$

$$+ \frac{\lambda_1}{4M_{Pl}} (N_1^4 + N_2^4 + N_3^4) + \frac{\lambda_2}{2M_{Pl}} (N_2^2 N_3^2 + N_3^2 N_1^2 + N_1^2 N_2^2)$$

$$
\begin{aligned}
V = & |m_N N_1 + f N_2 N_3 + \frac{\lambda_1}{M_{Pl}} N_1^3 + \frac{\lambda_2}{M_{Pl}} N_1 (N_2^2 + N_3^2)|^2 \\
& + |m_N N_2 + f N_3 N_1 + \frac{\lambda_1}{M_{Pl}} N_2^3 + \frac{\lambda_2}{M_{Pl}} N_2 (N_3^2 + N_1^2)|^2 \\
& + |m_N N_3 + f N_1 N_2 + \frac{\lambda_1}{M_{Pl}} N_3^3 + \frac{\lambda_2}{M_{Pl}} N_3 (N_1^2 + N_2^2)|^2 .
\end{aligned}
$$

The usual solution of $V = 0$ is $\langle N_{1,2,3} \rangle = 0$, but the following is also possible:

$$\langle N_{2,3} \rangle = 0, \quad \langle N_1 \rangle^2 = \frac{-m_N M_{Pl}}{\lambda_1}$$

$$A = -2m_N, \quad B = (1 - \lambda_2/\lambda_1) m_N, \quad C = f \langle N_1 \rangle .$$

The soft term $-h\langle N_1\rangle(\nu_1\phi^0 - l_1\phi^+)$ must also be added to allow (ν_1, l_1) and (ϕ^+, ϕ^0) to remain massless at the seesaw scale. The resulting theory is then protected below the seesaw scale by the usual $R-$parity of a supersymmetric theory.

Thus A_4 allows tribimaximal neutrino mixing to be generated automatically from the $N_{1,2,3}$ superfields themselves. However, the neutrino mass eigenvalues are not predicted.

Approximate Tribimaximal Mixing

Consider next the assignments of case (II), where $(\nu_i, l_i), l_i^c \sim \underline{3}$ and $(\phi_i^0, \phi_i^-) \sim \underline{1}, \underline{1}', \underline{1}''$,

then $\mathcal{M}_l$ is diagonal with

$$\begin{pmatrix} m_e \\ m_\mu \\ m_\tau \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} h_1 v_1 \\ h_2 v_2 \\ h_3 v_3 \end{pmatrix}.$$

Here $\mathcal{M}_\nu^{(e,\mu,\tau)} = \mathcal{M}_\nu$ already.

Let $d = e = f$, then

$$\mathcal{M}_\nu = \begin{pmatrix} a+b+c & d & d \\ d & a+\omega b+\omega^2 c & d \\ d & d & a+\omega^2 b+\omega c \end{pmatrix}$$

Assume $b = c$ and rotate to the basis $[\nu_e, (\nu_\mu + \nu_\tau)/\sqrt{2}, (-\nu_\mu + \nu_\tau)/\sqrt{2}]$, then

$$\mathcal{M}_\nu = \begin{pmatrix} a+2b & \sqrt{2}d & 0 \\ \sqrt{2}d & a-b+d & 0 \\ 0 & 0 & a-b-d \end{pmatrix},$$

i.e. maximal $\nu_\mu - \nu_\tau$ mixing and $U_{e3} = 0$.

We now have $\tan 2\theta_{12} = 2\sqrt{2}d/(d-3b)$
For $b << d$, $\tan 2\theta_{12} \to 2\sqrt{2}$, i.e. $\tan^2\theta_{12} = 1/2$,
but $\Delta m^2_{sol} << \Delta m^2_{atm}$ implies $2a + b + d \to 0$,
so that $\Delta m^2_{atm} \to 6bd \to 0$ as well.

Therefore, $b \neq 0$ is required, and $\tan^2\theta_{12} \neq 1/2$,
but should be close to it, because $b = 0$ enhances the symmetry of $\mathcal{M}_\nu$ from Z_2 to S_3.
Here $\tan^2\theta_{12} < 1/2$ implies inverted ordering
and $\tan^2\theta_{12} > 1/2$ implies normal ordering.

Models based on (II):

- Chen/Frigerio/Ma(2005):
  $3b = -ef/d - \omega^2 fd/e - \omega de/f,$
  $3c = -ef/d - \omega fd/e - \omega^2 de/f.$

- Ma(2005-2): two complicated equalities.

- Hirsch/Villanova/Valle/Ma(2005): $d = e = f,\ [b = c]$.

- Ma(2005-3): $b = c,\ e = f,\ (a + 2b)d^2 = (a - b)e^2$.

Particle content of models based on (II):

A_4(II)	(ϕ^+, ϕ^0)	N	(ξ^{++}, ξ^+, ξ^0)	χ^0
CFM05	$1, 1', 1''$	3	1	3
M05-2	1	3	3	$1, 1', 1'', 3$
HVVM05	$1, 1', 1''$	—	$1, 1', 1'', 3$	—
M05-3	$1, 1', 1'', 3$	$1, 1', 1''$	—	—

S_4 and B_4

Approximate tribimaximal mixing is obtained naturally in a supersymmetric seesaw model [Ma(2005-4)] with S_4, resulting in $b = c$ and $e = f$ [case (II)]:

$$\mathcal{M}_\nu = \begin{pmatrix} a+2b & e & e \\ e & a-b & d \\ e & d & a-b \end{pmatrix}$$

Here $b = 0$ and $e = d$ are related limits.

Permutation Symmetry S_4

The group of permutation of 4 objects is S_4. It contains both S_3 and A_4. It is also the symmetry group of the hexahedron (cube) and the octahedron.

class	n	h	χ_1	$\chi_{1'}$	χ_2	χ_3	$\chi_{3'}$
C_1	1	1	1	1	2	3	3
C_2	3	2	1	1	2	–1	–1
C_3	8	3	1	1	–1	0	0
C_4	6	4	1	–1	0	–1	1
C_5	6	2	1	–1	0	1	–1

Multiplication rule:

$$\underline{3}\times\underline{3} = \underline{1}(11+22+33)+\underline{2}(11+\omega^2 22+\omega 33,\, 11+\omega 22+\omega^2 33)$$

$$+\underline{3}(23+32,\, 31+13,\, 12+21)+\underline{3}'(23-32,\, 31-13,\, 12-21)$$

$$\underline{3}' \times \underline{3}' = \underline{1} + \underline{2} + \underline{3}_S + \underline{3}'_A$$

$$\underline{3} \times \underline{3}' = \underline{1}' + \underline{2} + \underline{3}'_S + \underline{3}_A$$

Note that both $\underline{3} \times \underline{3} \times \underline{3} = \underline{1}$ and $\underline{2} \times \underline{2} \times \underline{2} = \underline{1}$ are possible in S_4.

Let $(\nu_i, l_i), l_i^c, N_i \sim \underline{3}$.

Assume singlet Higgs superfields $\sigma_{1,2,3} \sim \underline{3}$ and $\zeta_{1,2} \sim \underline{2}$, then

$$\mathcal{M}_N = \begin{pmatrix} M_1 & h\langle\sigma_3\rangle & h\langle\sigma_2\rangle \\ h\langle\sigma_3\rangle & M_2 & h\langle\sigma_1\rangle \\ h\langle\sigma_2\rangle & h\langle\sigma_1\rangle & M_3 \end{pmatrix}$$

where

$M_1 = A + f(\langle\zeta_2\rangle + \langle\zeta_1\rangle)$
$M_2 = A + f(\omega\langle\zeta_2\rangle + \omega^2\langle\zeta_1\rangle)$
$M_3 = A + f(\omega^2\langle\zeta_2\rangle + \omega\langle\zeta_1\rangle)$

The most general S_4-invariant superpotential of σ and ζ

is given by
$W = M(\sigma_1\sigma_1 + \sigma_2\sigma_2 + \sigma_3\sigma_3) + \lambda\sigma_1\sigma_2\sigma_3 + m\zeta_1\zeta_2 + \rho(\zeta_1\zeta_1\zeta_1 + \zeta_2\zeta_2\zeta_2) + \kappa(\sigma_1\sigma_1 + \omega\sigma_2\sigma_2 + \omega^2\sigma_3\sigma_3)\zeta_2 + \kappa(\sigma_1\sigma_1 + \omega^2\sigma_2\sigma_2 + \omega\sigma_3\sigma_3)\zeta_1$

The resulting scalar potential has a minimum at $V = 0$ (thus preserving supersymmetry) only if $\langle\zeta_1\rangle = \langle\zeta_2\rangle$ and $\langle\sigma_2\rangle = \langle\sigma_3\rangle$, so that

$$\mathcal{M}_N = \begin{pmatrix} A+2B & E & E \\ E & A-B & D \\ E & D & A-B \end{pmatrix}$$

$\mathcal{M}_l$ is diagonal by choosing $\phi^l_{1,2,3} \sim \underline{1} + \underline{2}$.
$\mathcal{M}_{\nu N}$ is proportional to the identity by choosing $\phi^N_{1,2,3} \sim \underline{1} + \underline{2}$, with the latter having zero vacuum expectation value. Thus

$$\mathcal{M}_\nu = \begin{pmatrix} a+2b & e & e \\ e & a-b & d \\ e & d & a-b \end{pmatrix}$$

This avoids the *ad hoc* assumption of equating the contributions of the $\underline{1}'$ and $\underline{1}''$ contributions in A_4 models.

Exact tribimaximal mixing has also been obtained [Grimus/Lavoura(2005)] using the Coxeter group B_4, which is the symmetry group of the hyperoctahedron (4-crosspolytope) with 384 elements. Here $\mathcal{M}_l$ is diagonal with (ν_i, l_i), l_i^c, and (ϕ_i^+, ϕ_i^0) belonging to 3 different 3-dimensional representations of B_4, i.e.

$$a_1 b_1 c_1 + a_2 b_2 c_2 + a_3 b_3 c_3 = \underline{1}$$

The $\mathcal{M}_N$ of S_4 [M05-4] is then reduced by $D = E + 3B$.

Some Remarks

With the application of the non-Abelian discrete symmetry A_4, a plausible theoretical understanding of the HPS form of the neutrino mixing matrix has been achieved.

Another possibility is that $\tan^2\theta_{12}$ is not $1/2$ but close to it. This has theoretical support in an alternative version of A_4, but is much more natural in S_4.